$\boxed{\text{Omar}}$ ~~Stab(Coh X)~~

Stab (D^b Coh X) for X elliptic curve (smooth)
higher genus (smooth)

References: Atiyah, "Vec. Bundles on Ellip. curve"
Bridgeland, last 2 pages
Macri, "Some Examples of Stability Conditions"

Goal: Describe space of numerical stability conditions on an elliptic curve (or any smooth proj. curve of genus ≥ 1).

I'll describe action of $\widetilde{GL(2,\mathbb{R})}$ on Stab.

If you look at "numerical stab. cond." (those that factor through numerical Groth. group) the action will be free and transitive.

Are we missing something by factoring only through numerical Groth group?
Yes. I'll describe Coh(X) for elliptic curve

Coh(E) Atiyah classified vec bundles.

First, any $\mathcal{F} \in Coh(E)$ is the direct sum of torsion part and a vector bundle

$$\mathcal{F} \cong \mathcal{F}_{tor} \oplus \mathcal{F}/\mathcal{F}_{tor}$$

Not difficult theorem to get this direct sum

stalks are free, hence locally free by Nakayama

Given a module M, an element m is torsion if for some $a \in A$, a not zero divisor, $am = 0$.

$M_{tor} \subset M$.

Taking sheafification of torsion presheaf, stalks will be torsion.

Call $\mathcal{F}$ indecomposable if $\mathcal{F}$ is not $\oplus$ of two sheaves. So an indecomposables are either torsion, or ~~free~~ locally free

Atiyah did the case of locally free ones: Vec bundles. We need to talk about degrees.

<u>Degree of a coh sheaf on smooth curve</u>

$\mathcal{L}$ line bundle on X. $\quad \mathcal{L} \longleftrightarrow$ divisors $\sum n_i p_i \longleftrightarrow \sum n_i \in \mathbb{Z}$

$\mathcal{L} \cong \mathcal{O}(-D)$

pts on X

$n_i > 0 \Rightarrow$ zeroes @ p_i

degree of $\mathcal{L}$.

Zeroes-poles of rational section

Euler char:

$$\chi(\mathcal{F}) = \sum_i (-1)^i H^i(\mathcal{F})$$

Thm
$$\deg(\mathcal{L}) = \chi(\mathcal{L}) - \underbrace{\chi(\mathcal{O}_X)}_{= 1-g} \qquad \text{for line bundles}$$

g genus of X.
X sm proj curve

Proof by induction on $|D|$. Not hard, using additivity of χ wrt SES

Riemann-Roch Thm (using Serre duality)

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F})$$

$$\underset{\text{Serre duality}}{=} h^0(\mathcal{F}) - h^0(\mathcal{F}^\vee \otimes \omega_X)$$

$\mathcal{F}$ any coherent sheaf ~~vector bundle~~ ~~(needs to be locally free)~~. Since $h^1(\text{torsion}) = 0$

For $\mathcal{F}$ vec bundle, define

$$\deg(\mathcal{F}) = \underbrace{\chi(\mathcal{F}) - \operatorname{rk}\mathcal{F}}_{\text{additive on SES}} \cdot \chi(\mathcal{O}_X).$$

So degree is SES.

What is rk of a sheaf? dimension of stalk at generic point (over local ring @ generic pnt.)

$$\operatorname{rk}\mathcal{F} = \dim_{\mathcal{O}_{X,\eta}} \mathcal{F}_\eta \quad \leftarrow \text{generic pt}$$

For vec bundle $\mathcal{F}$, $\deg \mathcal{F} = \deg(\mathcal{F}^{\wedge \text{top}})$

↑ line bundle

Ex on $\mathbb{A}^1$,

$$0 \to \widetilde{k[t]} \xrightarrow{\times t} \widetilde{k[t]} \to \widetilde{k[t]/(t)} \to 0$$

Thm (Atiyah) Fix ellptc curve E.

$\forall$ (rk, deg) $= (r, d)$,

$$V_{r,d} := \left\{ \begin{array}{l} \text{indecomp. vec} \\ \text{bundles of} \\ \text{rank } r, \\ \text{degree } d \end{array} \right\} \cong E.$$

Ex For line bundles, Jacobian $(E) \cong E$.

The map

$$\begin{array}{ccc} V_{r,d} & \xrightarrow{\det} & V_{1,d} \\ \downarrow & \gcd(r,d) & \downarrow \\ E & \longrightarrow & E \\ x & \longmapsto & \gcd(r,d) \cdot x \end{array}$$

uses additive structure on E.

And given $E_{r,d} \in V_{r,d}$,

$$V_{r,d} \cong \{ E_{r,d} \otimes \mathcal{L}, \ \mathcal{L} \text{ degree } 0 \}.$$

In fact, $E_{r,d} \cong E_{r,d} \otimes \mathcal{L} \implies \mathcal{L}^r \cong \mathcal{O}_E$.

Ex

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \cdots \oplus \mathcal{L}^{\otimes (r-1)} \oplus \mathcal{O}_X$$

Then $\mathcal{E} \otimes \mathcal{L} \cong \mathcal{E}$, if $\mathcal{L}^{\otimes r} \cong \mathcal{O}_X$.

Based on Atiyah's thm,

$$K_0(D^b \mathrm{Coh}\, E) \cong \underset{\text{rank}}{\mathbb{Z}} \oplus \underset{\text{degree}}{\mathbb{Z}} \oplus \mathrm{Pic}^0 E$$

rk 1, deg 0

Def'n The numerical Grothendieck group

$$N(D^b \mathrm{Coh}\, X)$$

is

$$K_0(D^b \mathrm{Coh}\, X) / \mathrm{Ker}\langle\ ,\ \rangle.$$

Sometimes, left kernel $\neq$ right kernel.

Defn Mukai pairing

$$\langle \mathcal{E}, \mathcal{F} \rangle = \sum (-1)^i \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$$

$$= \sum (-1)^i \dim \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i]).$$

This is bilinear on K_0.

Left kernel $= \{\mathcal{E} \mid \langle \mathcal{E}, \mathcal{F} \rangle = 0\ \forall \mathcal{F}\}$.

Setting $S(\mathcal{E}) = \mathcal{E} \otimes \omega_X[\dim X]$,

$\mathrm{Hom}(A,B) \cong$ ~~$\mathrm{Hom}(B,A)^\vee$~~ $\cong \mathrm{Hom}(B, SA)^\vee$.

So $\langle \mathcal{E}, \mathcal{F} \rangle \cong \langle \mathcal{F}, S(\mathcal{E}) \rangle$.

For a curve,

$-\langle \mathcal{E}, \mathcal{F} \rangle \cong \langle \mathcal{F}, \mathcal{E} \otimes \omega_X \rangle$

Since E is Calabi-Yau,

$-\langle \mathcal{E}, \mathcal{F} \rangle = \langle \mathcal{F}, \mathcal{E} \rangle$

$\boxed{\text{So left kernel} = \text{right kernel!}}$

Ex $N(D^b \mathrm{Coh}\, E) \cong \underset{\text{rk}}{\mathbb{Z}} \oplus \underset{\text{deg}}{\mathbb{Z}}$

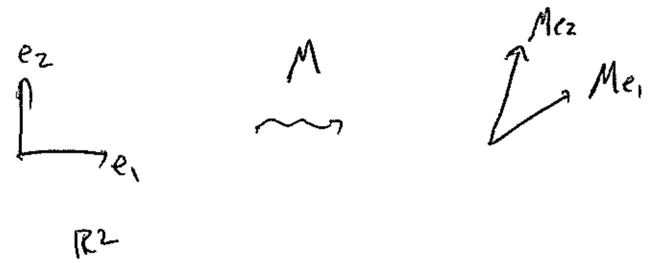
By a numerical stability condition, we mean those (Z, P) s.t.

$$Z : K_0 \longrightarrow \mathbb{C}$$

factors through $N(D^b \mathrm{Coh}\, E)$.

Action of $\widetilde{GL^+(2,\mathbb{R})}$ on Stab

Well, $GL^+(2,\mathbb{R}) \underset{\text{hom. eq.}}{\simeq} SO(2) \cong S^1$.

Given 

$Me_1 \neq 0$,

$Me_2 \in H_{Me_1} =$ 

$\{\text{Choice of } Me_1\} \simeq S^1$, and H_{Me_1} is contractible.

You can think of $\widetilde{GL^+(2,\mathbb{R})}$ (the univ. cover of $GL^+(2,\mathbb{R})$) as

$$\widetilde{GL^+(2,\mathbb{R})} = \{ (M,f) \mid M \in GL^+(2,\mathbb{R}), \; f : \mathbb{R} \to \mathbb{R},\; f(t+1) = f(t)+1, \; Me^{i\theta} \in \mathbb{R}_{>0}\, e^{if(\theta)\pi} \}$$

So $\dfrac{Me^{i\pi\theta}}{|Me^{i\pi\theta}|} \in S^1$, $Me^{i\pi\theta}$ gives path in $\mathbb{R}^2 \setminus \{0,0\}$, hence in S^1!

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "\frac{Me^{i\pi\theta}}{|Me^{i\pi\theta}|}"'] & S^1 \end{tikzcd}$$

The action

$$\widetilde{GL^+(2,\mathbb{R})} \circlearrowright \mathrm{Stab}$$

is given by

$$(M, f)\cdot(Z, P) = (M^{-1}\circ Z,\ P\circ f).$$

There's also an action of the group of exact equivalences of your derived category.

$$\mathrm{Aut}(D^b \mathrm{Coh}) \circlearrowright \mathrm{Stab}$$

by

$$\Phi\cdot(Z, P) = (Z\circ\Phi^{-1},\ \Phi P).$$

These two actions commute.

<u>Back to curves</u>. Let X be sm proj curve, genus >0.
Let $\mathrm{Stab}(X)$ be space of numerical Groth grp. Then

$$\mathrm{Stab}(X) \cong \widetilde{GL^+(2,\mathbb{R})}\cdot\sigma$$

where $\sigma(E) = -\deg E + i\,\mathrm{rk}\,E$.

We present proof due to Macri.

Note: $\mathrm{Coh}X$ is of homological dimension 1.

That is, $\mathrm{Ext}^{\geq n} = 0 \quad \forall\, n \geq 2$.

This is also a consequence of Serre duality:

$$\mathrm{Ext}^2(E,F) = \mathrm{Ext}^{-1}(F, E \otimes \omega_X) = 0$$

On X a curve, sm projective, every $E \in D^b\mathrm{Coh}X$ is

$$E \simeq \bigoplus \mathcal{H}^i(E)[-i]$$

cohomology sheaves.

(We in genl have a H-N-like filtration given by Coh(X) being a heart; over a curve, it splits.) Why?

$$\longrightarrow E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \longrightarrow$$

$$\begin{array}{ccccccccc} E^{i-1} & \to & E^i & \to & \ker d^{i+1} & \to & 0 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & & & \\ E^{i-1} & \to & E^i & \to & E^{i+1} & \to & \operatorname{image} d^{i+1} & \to & 0 \to \cdots \end{array}$$

two equivalent truncations, $\tau_{\leq i+1}$.

Truncating one step before, $\tau_i \longrightarrow \tau_{i+1}$

$$\begin{array}{ccc} E^i & \to & \operatorname{im} d^i \\ \downarrow & & \downarrow \\ E^i & \to & \ker d^{i+1} \end{array} \longrightarrow \mathcal{H}^{i+1}(E) \to 0$$

So we get SES of complexes

$$\tau_i \xi \to \tau_{i+1}\xi \to \mathcal{H}^{i+1}(\xi)[-i-1] \xrightarrow{+1}$$

exact triangle.

By induction, assume

$$\tau_i \xi \cong \bigoplus_{j\le i} \mathcal{H}^j(\xi)[-j].$$

Now we need to show

$$\mathrm{Ext}^1\Big(\mathcal{H}^{i+1}(\xi)[-i-1], \bigoplus_{j\le i}\mathcal{H}^j(\xi)[-j]\Big)$$

is zero. i.e.,

$$\hom\Big(\mathcal{H}^{i+1}(\xi)[-i-1], \bigoplus_{j\le i}\mathcal{H}^j(\xi)[-j+1]\Big)$$

$$\cong \bigoplus_{j\le i} \hom\big(\mathcal{H}^{i+1}(\xi), \mathcal{H}^j(\xi)[2+i-j]\big)$$

$[_{\geq 2}$.

$\cong 0$. (Using the fact we're on a curve!)

So things split!

Lemma (1) (Z, P) a numerical stability condition on $D^b\mathrm{Coh}(X)$

Then all skyscraper sheaves k_x and all line bundles are σ-stable.

To prove this, we use

Lemma (2) (Gorodentsev, Kuleshov, Rudakov) Fix X, genus ≥ 1.

Given an exact triangle

$$A \to E \to B$$

where E is in the heart $\mathrm{Coh}(X)$,

and $\mathrm{Hom}^{\leq 0}(A, B) = \mathrm{Ext}^{\leq 0}(A, B) = 0$,

then A, B are also in the heart. $(\mathrm{Coh}\,X)$.

Milder Lemma (3): Given $A \to E \to B$ triangle, $E \in \mathrm{Coh}(X)$

Then $\mathrm{Ext}^{<0}(A, B) = 0 \Longrightarrow A \cong A_0 \oplus A_1[-1]$

and $B \cong B_0 \oplus B_{-1}[1]$.

using that $\mathrm{Coh}(X)$ has homological dimension 1

Then, in positive genus case, w/ $\hom^0(A, B) = 0$, we get result.

Pf of 3

$$A \cong \bigoplus A_i[-i]$$

$$B \cong \bigoplus B_j[-j]$$

Since E is in heart, LES of H° looks like

$$0 \to B_{-1} \to A_0 \to E \to B_0 \to A_1 \to 0 \quad \text{exact,}$$

and

$$B_{i-1} \cong A_i \quad \forall \; i \neq 0, 1.$$

(Since $E_i = 0 \; \forall \; i \neq 0$, by $E \in Coh(X)$).

If $A_i \neq 0$ for some $i \neq 0, 1$, then

$$\hom(A_i, B_{i-1}) \neq 0$$

But $\hom(A_i, B_{i-1}) \cong Ext^{-1}(A_i[-i], B_{i-1}[-i+1])$

$= 0$ since $Ext^{-1}(A, B) = 0$.

So $B_{i-1} \cong A_i \cong 0$.

More next time!

Omar, Part II

Th. Oct 24, 2013

Thm X sm, projective curve.

Genus >0.

Then

$$\mathrm{Stab}(X) \cong \widetilde{GL}^+(2,\mathbb{R})$$

numerical stability conditions

Lemma $\forall$ numerical stab cond,

line bundles + skyscraper sheaves are all stable.

Recall: $\mathrm{Coh}(X)$ has homological dimension 1. So every object is quis to $\oplus$ of cohom. objects

Sublemma Let $A \to E \to B$ triangle in $D^b\mathrm{Coh}X$, w/ $E \in \mathrm{Coh}X$ and $\mathrm{Ext}^{<0}(A,B)=0$, then

$A \cong A_0 \oplus A_1[-1]$ and $B \cong B_0 \oplus B_{-1}[1]$.

(Cohomologies in $0,1$ and $0,-1$.)

How'd the proof go? LES:

$$0 \to B_{-1} \to A_0 \to E \to B_0 \to A_1 \to 0.$$

And: $A_i \cong B_{i-1}$ $\forall$ other i, but $\mathrm{Ext}^{<0}=0$, so $A_i \cong B_{i-1} \cong 0$.

Sublemma With same hypotheses as before,

w/ $g(X) > 0$, and $\mathrm{Ext}^{\leq 0}(A,B) = 0$

genus — no hom $= \mathrm{Ext}^0$ either

then $A, B \in \mathrm{Coh}(X)$.

This is where $g > 0$ appears

Pf If $B_{-1} \to A_0$ non-zero,

we have non-zero

$$B_{-1} \to A_0 \otimes \omega_X$$

ω_X has sections in positive genus, so tensor by a section.

By Serre duality,

this is a non-zero element in

$$\mathrm{Ext}^1(A_0, B_{-1}) = \mathrm{Ext}^1(A, B[-1])$$

so

$$\mathrm{Ext}^1(A_0, B_{-1}) \neq 0 \Rightarrow \mathrm{Ext}^0(A,B) \neq 0.$$

Dually $B_0 \to A_1$ must be zero. //

Now, on elliptic curve $\langle \xi, \mathcal{F} \rangle = \sum (-1)^i \dim \mathrm{Ext}^i(\xi, \mathcal{F})$

$$= -\deg \xi \, \mathrm{rk} \mathcal{F} + \deg \mathcal{F} \, \mathrm{rk} \xi.$$

(Why? $\mathrm{Ext}^{0,1}$ are only non-zero groups. First note its true for line bundles, or locally free guys)

Why? $\langle \xi, \mathcal{F} \rangle = ?$ $\quad \mathrm{Ext}^i(\xi, \mathcal{F}) = \mathrm{Ext}^i(\mathcal{O}, \xi^\vee \otimes \mathcal{F})$ since tensoring w/ locally free things is exact

if locally free $\langle \xi, \mathcal{F} \rangle = \chi(\xi^\vee \otimes \mathcal{F})$

$= \deg(\xi^\vee \otimes \mathcal{F}) + \mathrm{rk}\, \xi \cdot \mathrm{rk} \mathcal{F} \cdot \chi(\mathcal{O}_X)$. If $\xi^\vee, \mathcal{F}$ line bundles, $\deg \xi^\vee \otimes \mathcal{F} = \deg \mathcal{F} - \deg \xi^\vee$

On elliptic curve,

$$\langle \xi, \mathcal{F} \rangle = \deg \mathcal{F} \,\mathrm{rk}\, \xi - \deg \xi \,\mathrm{rk}\, \mathcal{F}.$$

In general,

$$\langle \xi, \mathcal{F} \rangle = \deg \mathcal{F} \,\mathrm{rk}\, \xi - \deg \xi \,\mathrm{rk}\, \mathcal{F} + \mathrm{rk}\, \xi \,\mathrm{rk}\, \mathcal{F} (1-g).$$

Since true for line bundles;
we extend to all K_0 since line bundles generate K_0.

Now lets prove line bundles + skyscrapers are stable.

Pf. First, semistability. Use sublemma.

Let $E = \mathcal{L}$ or k_x. Given H-N filt,

examine $A = A_0 \longrightarrow E$.

Then let
$$\begin{tikzcd} A \arrow[r] & E \arrow[dl] \\ B \arrow[u, "+1"] & \end{tikzcd}$$
be Δ.

Then we have no non-~~negative~~ positive ext btwn A and B. $\mathrm{Ext}^{\leq 0}(A,B)=0$.

By Lemma $A, B \in \mathrm{Coh}(X)$.

So this is SES

If $E = k_x$, has no subobjects! Done.

If $E = \mathcal{L}$, A must also be line bundle.
But then B is torsion, this contradicts
$\mathrm{Ext}^{0} = 0$.

Pf of stability.

Use Jordan-Holder filtration in slice that E lives in.

$$E \in \mathcal{P}(\phi), \quad \phi = \phi(E)$$

Recall each slice $\mathcal{P}(\phi)$ is abelian.

It's finite length since $\sigma \in$ Stab is locally finite.

So take JH filtration

$$E_0 \to \cdots \to E_n = E,$$

each E_i / E_{i-1} stable of phase ϕ.

$$\begin{array}{c} E_i/E_{i-1} \\ \parallel \\ S_i \end{array}$$

If $\mathrm{Hom}(S_i, E) \neq 0$, examine

$$\{E' \subset E \mid \text{JH fact. of } E' \text{ are } \cong S_i\} \ni 0, S_i.$$

But let E' be maximal.

Take $0 \to E' \to E \to E'' \to 0$, So

$\mathrm{hom}(E', E'') = 0$ since E'' has no subobject S_i.

Now, take JH of E' and of E''

Then you can put all JH factors $\cong$ to S_i at beginning of JH filt of E.

If E is skyscraper, $0 \to E' \to E \to E'' \to 0$

$\Rightarrow E' = 0$, done. No stable factors at all

$(E' = 0 \ \forall S_i)$

If E line bundle, similar arg from before $\Rightarrow E'$ line bundle, E'' torsion

$\Rightarrow \mathrm{hom}(E', E'') \neq 0 \ \rightarrow\leftarrow$

if $\mathcal{P}(\phi) \subset \mathrm{Coh}(X)$, and if it DOESN'T contain other complexes

Now, we need to verify $P(\phi) \subset Coh(X) \subset D^b Coh(X)$

But $hom(E', E'') = 0$, so $Ext^{\leq 0}(E', E'') = 0$ since $\phi(E') = \phi(E'')$

$$\left(Ext^{-n}(E', E'') = Ext(E', E''[-n])\right.$$
$$= 0 \text{ since } \phi - n < \phi.$$

Rmk If $E'' = 0$,
we see $E' \cong E$
while $E' \ncong S_i$.
So E is stable. //

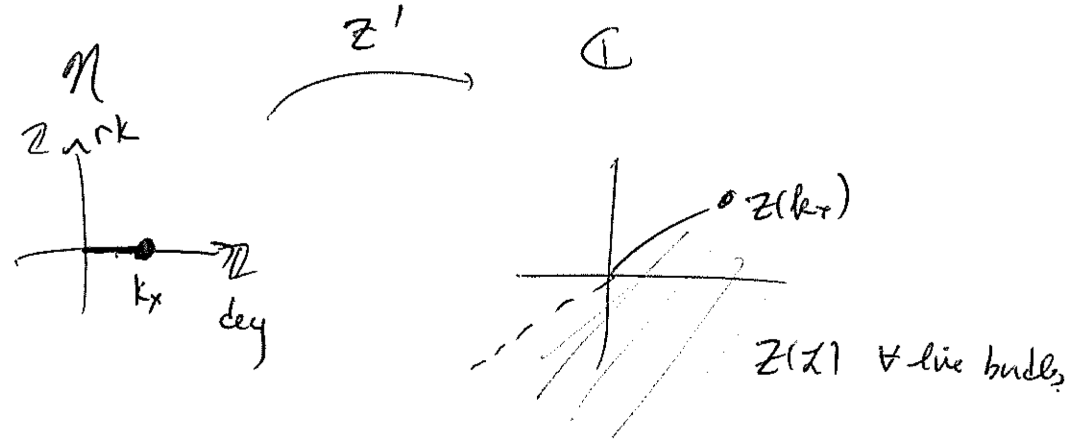
Now, $\forall \sigma' \in Stab(X)$
$hom(\mathcal{L}', k_x) \neq 0$, while $\mathcal{L}', k_x$ are stable, $\mathcal{L}'$ line bundle

$\Longrightarrow$ $ext^1(k_x, \mathcal{L}) = hom(\mathcal{L} \otimes \omega_X^{-1}, k_x) \neq 0$, $(\mathcal{L}' = \mathcal{L} \otimes \omega_x^{-1})$.

So $\phi(\mathcal{L}) < \phi(k_x)$ $\forall$ line bundles since $\mathcal{L}, k_x$ are stable NOT canonically

Also, $\phi(k_x) - 1 < \phi(\mathcal{L})$ since $ext^1(k_x, \mathcal{L}) \neq 0$

This already shows



So $Z' : \mathcal{N} \otimes \mathbb{R} \longrightarrow \mathbb{R}^2$ is an $\cong$.

This shows action must be free. Why? We recover an element $GL(2, \mathbb{R})$
by comparing Z and Z'. Then recover real fn ~~f~~ f by ~~f~~ comparing $P(\mathcal{L}), P(\mathcal{L})$

Why transition? Given σ, σ' find appropriate matrix taking Z to Z',

~~[illegible]~~ then compensate phase shifts by f.

Now $P(\Phi(k_x - 1), \Phi(k_x)]$ contains all Z and all k_x.

So $P(\Phi(k_x - 1), \Phi(k_x)]$ is standard heart. //