Numerical stability conditions on curves of positive genus

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Our goal is to prove the following theorem:

Theorem. *Let X be a smooth projective curve (over the complex numbers). Then the action of $\widetilde{GL^+}(2,\mathbb{R})$ on the moduli space $\mathrm{Stab}(X)$ of numerical stability conditions on X is free and transitive.*

This theorem is due to Bridgeland [1] for the case of an elliptic curve, and to Macrì [2] for all positive genera. Before jumping in to the proof, we'll define the $\widetilde{GL^+}(2,\mathbb{R})$ action and the numerical Grothendieck group of $\mathrm{D}^b\mathrm{Coh}(X)$.

The $\widetilde{GL^+}(2,\mathbb{R})$ action on stability conditions

To describe $\widetilde{GL^+}(2,\mathbb{R})$, let's first look at the homotopy type of $GL^+(2,\mathbb{R})$. Given any non-singular 2×2 matrix, we can perform Gram-Schmidt orthonormalization on it to get an orthogonal matrix, and furthermore, looking at the formulas for the Gram-Schmidt process, we see we can "do it gradually", obtaining a deformation retraction of $\widetilde{GL^+}(2,\mathbb{R})$ onto $SO(2,\mathbb{R})$. Now, $SO(2,\mathbb{R})$ just consists of rotations, which are parametrized by an angle in S^1, so $GL^+(2,\mathbb{R})$ is homotopy equivalent to S^1, and since the Gram-Schmidt process keeps the direction of the first column of the matrix, this angle is really given by the phase of M applied to the unit vector $(1,0)$. The universal cover of $GL^+(2,\mathbb{R})$ will be a $\mathbb{Z}$-cover of $GL^+(2,\mathbb{R})$, where we "unwind" that S^1, which we can describe as follows:

$$\widetilde{GL^+}(2,\mathbb{R}) \cong \{(M,f) : M \in GL^+(2,\mathbb{R}), f : \mathbb{R} \to \mathbb{R},$$
$$f(x+1) = f(x)+1, \forall\theta\ Me^{i\pi\theta} \in \mathbb{R}_+ e^{i\pi f(\theta)}\}$$

(Here we have identified $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way, to be able to apply M to $e^{i\pi\theta}$.) The group operation is given by $(M,f)\circ(M',f') = (MM', f\circ f')$. And, as a check, the fiber above a given matrix M consists of all lifts f such that

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle e^{i\pi t}}\downarrow & & \downarrow{\scriptstyle e^{i\pi t}} \\ S^1 & \xrightarrow{\hat{M}} & S^1 \end{array}$$

where $\hat{M}v = \frac{Mv}{|Mv|}$. There are only a $\mathbb{Z}$'s worth of such lifts, each being determined by $f(0)$.

Given a stability condition $\sigma = (Z, \mathcal{P})$ we set $(M,f)\cdot\sigma = (M^{-1}\circ Z, \mathcal{P}\circ f)$. This determines a right action of $\widetilde{GL^+}(2,\mathbb{R})$ on stability conditions that corresponds to the intuitive notion of "rotating a stability condition" that we have discussed in the course. Recall, in particular, that rotating a stability condition a full 2π is supposed to shift the heart by 2.

There is also a left action of the triangulated auto-equivalences of $\mathrm{D}^b\mathrm{Coh}(X)$ on stability conditions given by $\alpha \in \mathrm{Aut}(\mathrm{D}^b\mathrm{Coh}(X))$ acting as $\alpha\cdot(Z,\mathcal{P}) = (Z\circ\alpha^{-1}, \alpha(\mathcal{P}))$. This action commutes with the $\widetilde{GL^+}(2,\mathbb{R})$ action, and we will have nothing more to say about it.

The numerical Grothendieck group

The *Euler form* or *Mukai pairing* on $K_0(\mathrm{D}^b\mathrm{Coh}(X))$ is defined by the following formula:
$$\langle\mathcal{E},\mathcal{F}\rangle := \sum_{\imath\in\mathbb{Z}} (-1)^i \dim \mathrm{Ext}^i(\mathcal{E},\mathcal{F}).$$

Given an exact triangle $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3$ we can easily check that $\langle\mathcal{E}_1,\mathcal{F}\rangle - \langle\mathcal{E}_2,\mathcal{F}\rangle + \langle\mathcal{E}_3,\mathcal{F}\rangle = 0$ by using the long exact sequence relating the $\mathrm{Ext}^*(\mathcal{E}_i,\mathcal{F})$ for $i = 1,2,3$. This, with an analogous argument for $\mathcal{F}$, shows that the

Mukai pairing indeed descends to the Grothendieck group. Notice also that $\langle \mathcal{E}[n], \mathcal{F}\rangle = \langle \mathcal{E}, \mathcal{F}[n]\rangle = (-1)^n \langle \mathcal{E}, \mathcal{F}\rangle$.

Given any pairing we can consider its left and right kernels:

- The left kernel, $\ker^L := \{\mathcal{E} \in \mathrm{D}^b\mathrm{Coh}(X) : \text{for all } \mathcal{F}, \langle \mathcal{E}, \mathcal{F}\rangle = 0\}$, and
- The right kernel, $\ker^R := \{\mathcal{F} \in \mathrm{D}^b\mathrm{Coh}(X) : \text{for all } \mathcal{E}, \langle \mathcal{E}, \mathcal{F}\rangle = 0\}$.

Serre duality, when available, let's us relate these. Recall that in terms of a Serre functor S, Serre duality can simply be stated as a natural isomorphism $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, S(\mathcal{E}))^\vee$, which by shifting gives an analogous statement for Ext's and proves that $\langle \mathcal{E}, \mathcal{F}\rangle = \langle \mathcal{F}, S(\mathcal{E})\rangle$. This shows that the Serre functor gives an isomorphism between the left and right kernels. In case X is a smooth projective variety of dimension n, the Serre functor is given by $S(\mathcal{E}) = \mathcal{E} \otimes \omega_X[n]$, where ω_X is the sheaf of differential n-forms. In particular, if $\omega_X \cong \mathcal{O}_X$, such as for an elliptic curve or a Calabi-Yau variety, we get that $\ker^L = \ker^R$.

The *numerical Grothendieck group*, denoted $\mathcal{N}(X)$, is defined to be the quotient $K_0(\mathrm{D}^b\mathrm{Coh}(X))/\ker^L$, and a stability condition $(Z, \mathcal{P})$ is said to be numerical if the central charge Z factors through $\mathcal{N}(X)$.

In the case of a smooth projective curve of genus g, we have a simpler expression for the Mukai pairing. This expression involves the notions of rank and degree, which we will first define for coherent sheaves and then extend to homomorphisms on the Grothendieck group of $\mathrm{D}^b\mathrm{Coh}(X)$.

Rank. For a locally free sheaf we know what we want the rank to be: the rank of a stalk, all of which are free modules of the same rank. Any coherent sheaf on a smooth projective curve can be written as a direct sum of a locally free sheaf and a torsion sheaf[1] and we can define the rank to be simply the rank of the locally free part. Alternatively, we can define the rank to be the dimension of the stalk *at the generic point*; this definition makes it clear that the degree is additive on short exact sequences and thus defines a homomorphism $K_0(\mathrm{D}^b\mathrm{Coh}(X)) \to \mathbb{Z}$. (Notice that the operations of taking stalks at any point is exact, but

[1]This can be seen as follows: the classification theorem for finitely generated modules over a PID tells us that any coherent sheaf can be written as a direct sum of a torsion sheaf and a torsion-free sheaf; moreover, it tells us that the torsion-free part has free stalks and then Nakayama's lemma implies said part is locally free. Notice that because of the reliance on the classification theorem for PIDs, this is really special to curves.

here we need to use the generic point specifically to avoid the torsion part of the sheaf.)

Degree. For a line bundle L one can define the degree as follows: given any rational section σ of the line bundle, the degree of L is the number of zeros of σ minus the number of poles (both counted with the appropriate multiplicities); one can show this is independent of the chosen section σ. For this classical notion of degree, one has the following result, which is a(n easy) version of the Riemann-Roch theorem: $\deg L = \chi(L) - \chi(O_X)$; where $\chi(E) = \sum_i (-1)^i \dim H^i(E)$ is the Euler characteristic of the sheaf E. Generalizing that last formula, we will define the degree of an arbitrary coherent sheaf to be $\deg E = \chi(E) - (\operatorname{rank} E)\chi(O_X)$. This is defined as a linear combination of χ and rank, and is thus additive on short exact sequences, as required to pass to the Grothendieck group.

I now claim that for the Mukai pairing on a smooth projective curve of genus g, we have the formula

$$\langle E, F\rangle = -\deg E \operatorname{rank} F + \operatorname{rank} E \deg F - (1-g)\operatorname{rank} E \operatorname{rank} F.$$

Indeed, both sides are additive on short exact sequences in each variable separately and thus descend to bilinear forms on the Grothendieck group. This means to check they are equal it is enough to check this true for line bundles, as they generate the Grothendieck group. Even if we merely assume E is locally free, the identity to check simplifies a lot: since then $\operatorname{Ext}^i(E, F)$ is just $H^i(E^\vee \otimes F)$, the left-hand side becomes $\chi(E^\vee \otimes F)$, while the right-hand side, for locally free E reduces to $\deg(E^\vee \otimes F) - \operatorname{rank}(E^\vee \otimes F)\chi(\mathcal{O}_X)$. So for locally free E the claimed identity reduces to Riemann-Roch.

This formula shows that the morphism $\mathcal{N}(X) \to \mathbb{Z}^2$ given by the formula $E \mapsto (\operatorname{rank} E, \deg E)$ is an isomorphism: the formula above shows that the kernel of the Mukai pairing consists precisely of objects $E \in \mathrm{D}^b\mathrm{Coh}(X)$ such that $\deg E = \operatorname{rank} E = 0$.

The space of numerical stability conditions

Theorem. *Let X be a smooth projective curve (over the complex numbers). Then the action of $\widetilde{GL^+}(2, \mathbb{R})$ on the moduli space $\mathrm{Stab}(X)$ of numerical stability conditions on X is free and transitive.*

Throughout this section, X will always be some fixed smooth projective curve of positive genus. This is implicitly part of the hypothesis of every lemma here.

Most of work of proving the theorem will be in proving the following:

Lemma. *Line bundles and skyscraper sheaves are stable for any numerical stability condition σ on $\mathrm{D}^b\mathrm{Coh}(X)$.*

This in turn relies on the following result of Gorodentsev, Kuleshov, and Rudakov [3]:

Lemma. *If $A \to E \to B$ is a distinguished triangle in $\mathrm{D}^b\mathrm{Coh}(X)$ such that $\mathrm{Ext}^{\leq 0}(A, B) = 0$ and $E \in \mathrm{Coh}(X)$, then we must have $A, B \in \mathrm{Coh}(X)$ as well.*

Proof. We need the following basic fact about curves: the category $\mathrm{Coh}(X)$ for a smooth projective curve X has homological dimension 1, this means that $\mathrm{Ext}^k(U, V) = 0$ for all $k \geq 2$ and all $U, V \in \mathrm{Coh}(X)$. (It is in fact true, more generally, that for a smooth projective variety Y of dimension n, $\mathrm{Coh}(Y)$ has homological dimension n.)

Claim. *For any Abelian category $\mathcal{A}$ with homological dimension 1, we have that objects E of the bounded derived category $\mathrm{D}^b\mathcal{A}$ are direct sums of their homology objects, $E \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(E)$.*

(In the case $\mathcal{A} = \mathrm{Coh}(X)$, the object E can be thought of as a chain complex of coherent sheaves, and the $\mathcal{H}^i(X)$ are the cohomology sheaves of the complex, which one shouldn't confuse with the sheaf cohomology *groups* of E.)

Proof of claim. Recall the two versions of truncation of a complex $E = \cdots E^i \xrightarrow{d^i} E^{i+1} \to \cdots$ at degree i:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & E_{i-1} & \longrightarrow & \ker(d^i) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E_{i-1} & \longrightarrow & E_i & \longrightarrow & \mathrm{im}(d^i) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Each row has the following properties: (a) it is a subcomplex of E, (b) the inclusion into E gives an isomorphism on homology up to degree i, (c) its homology in degrees $i + 1$ and higher vanishes. The above diagram also depicts the inclusion of the top row into the bottom, and this is clearly a quasi-isomorphism. We will call the object of $\mathrm{D}^b\mathcal{A}$ corresponding to either row $\tau_{\leq i}E$.

Now consider the following short exact sequence of complexes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & E_{i-2} & \longrightarrow & E_{i-1} & \longrightarrow & \mathrm{im}(d^{i-1}) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E_{i-2} & \longrightarrow & E_{i-1} & \longrightarrow & \ker(d^i) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{H}^i(E) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

This gives rise to a distinguished triangle $\tau_{i-1}E \to \tau_i E \to \mathcal{H}^i(E)[-i]$.

Using this we can prove the claim by induction on the number of non-zero homology objects of a complex $E \in \mathrm{D}^b\mathcal{A}$. Let i be large enough that $\tau_i E = E$. Then we have a distinguished triangle $\tau_{i-1} \to E \to \mathcal{H}^i(E)[-i] \xrightarrow{f} \tau_{i-1}E[1]$. By the induction hypothesis, we have $\tau_{i-1}E \cong \bigoplus_{j<i} \mathcal{H}^j(E)[-j]$ and the morphism f therefore lies in:

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}^b\mathcal{A}}(\mathcal{H}^i(E)[-i], \bigoplus_{j<i} \mathcal{H}^j(E)[-j+1]) = \\
&\bigoplus_{j<i} \mathrm{Hom}_{\mathrm{D}^b\mathcal{A}}(\mathcal{H}^i(E)[-i], \mathcal{H}^j(E)[-j+1]) = \\
&\quad \bigoplus_{j<i} \mathrm{Hom}_{\mathrm{D}^b\mathcal{A}}(\mathcal{H}^i(E), \mathcal{H}^j(E)[i-j+1]),
\end{aligned}$$

but all of those groups are 0 since $i - j + 1 \geq 2$. This means that f must be 0 showing that the distinguished triangle splits as a direct sum and thus $E \cong \bigoplus_{j\leq i} \mathcal{H}^j(E)[-j]$. □

We'll use this to establish the part of the lemma that doesn't use that the genus is positive:

Claim. *If $A \to E \to B$ is a distinguished triangle in $\mathrm{D}^b\mathrm{Coh}(X)$ such that $\mathrm{Ext}^{<0}(A, B) = 0$ and $E \in \mathrm{Coh}(X)$, then each of A and B is concentrated in two degrees (not the same two degrees, though): $A \cong A_0 \oplus A_1[-1]$ and $B \cong B_0 \oplus B_{-1}[1]$.*

Proof of claim. By the previous claim we can write $A \cong \bigoplus A_i[-i]$ and $B \cong \bigoplus B_i[-i]$. The long exact sequence on homology induced by the triangle $A \to E \to B$ is very simple since E is concentrated in a single degree: for $i \neq 0, 1$, we get $A_i \cong B_{i-1}$, and we have an exact sequence

$$0 \to B_{-1} \to A_0 \to E \to B_0 \to A_1 \to 0.$$

If for some $i \neq 0, 1$ we had $A_i \neq 0$, then the isomorphism $A_i \cong B_{i-1}$ would give an element of $\mathrm{Hom}(A_i[-i], B_{i-1}[-i]) \cong \mathrm{Ext}^{-1}(A_i[-i], B_{i-1}[-(i-1)]) \subseteq \mathrm{Ext}^{-1}(A, B)$, which contradicts the assumptions. □

Now we use that in positive genus the canonical bundle ω_X has non-zero sections to conclude the proof of the lemma. Since the statement is now more than a page behind, I'll remind you that we have a distinguished triangle $A \to E \to B$ with $E \in \mathrm{D}^b\mathrm{Coh}(X)$ and $\mathrm{Ext}^{\leq 0}(A, B) = 0$ and want to prove that $A, B \in \mathrm{Coh}(X)$ too. By the above claim, we just need to show $A_1 = B_{-1} = 0$. We'll use the medium length exact sequence from the proof of the previous claim. To show that B_{-1} is 0 it is enough to show that the map $B_{-1} \to A_0$ in the sequence is 0. If it were not 0, twisting it by a section of ω_X would give a non-zero map $B_{-1} \to A_0 \otimes \omega_X$ and thus, by Serre duality, we'd have $0 \neq \mathrm{Ext}^1(A_0, B_{-1}) \subseteq \mathrm{Ext}^1(A, B[-1]) = \mathrm{Ext}^0(A, B)$, which is a contradiction. The proof of $A_1 = 0$ is similar. □

With this homological algebra out of the way we can now prove the geometric lemma we need, which I'll restate for convenience:

Lemma. *Line bundles and skyscraper sheaves are stable for any numerical stability condition $\sigma = (Z, \mathcal{P})$ on $\mathrm{D}^b\mathrm{Coh}(X)$.*

Proof. Let's first prove that line bundles and skyscraper sheaves are semistable. Let E be either one, and consider its Harder-Narasimhan filtration, let A be the first semistable factor in it, and form a distinguished triangle $A \to E \to B$. By semistability, we have $\mathrm{Ext}^{\leq 0}(A, B) = 0$, and the homological algebra lemma tells us that A and B are also just sheaves. This means that the distinguished triangle $A \to E \to B$ is just a short exact sequence $0 \to A \to E \to B \to 0$ in $\mathrm{Coh}(X)$. In the case E is a skyscraper sheaf this already forces $B = 0$, since it has no proper subobjects. If E is a line bundle, then A must also be a line bundle and the quotient, B, must be a torsion sheaf. But then, if $B \neq 0$, we'd have $\mathrm{Hom}(A, B) \neq 0$, too. Therefore $B = 0$ and E is semistable.

Now let's prove that they are even stable. Consider the Abelian category $\mathcal{P}(\phi(E))$ whose simple objects are the σ-stable objects of phase $\phi(E)$. If E is not stable, there will be some stable $S \in \mathcal{P}(\phi(E))$ such that $\mathrm{Hom}(S, E) \neq 0$. Now consider the set of subobjects of E all of whose filtration quotients in the Jordan-Hölder filtration are isomorphic to S, and let A be a maximal such subobject. Form the short exact sequence $0 \to A \to E \to B \to 0$. By maximality of A, B has no subobject isomorphic to S and since S is simple, $\mathrm{Hom}(S, B) = 0$. Since A is made out of extensions of copies of S, we also have $\mathrm{Hom}(A, B) = 0$, and we have $\mathrm{Ext}^{<0}(A, B) = 0$ just by semistability. So by the lemma, we again have $A, B \in \mathrm{Coh}(X)$. Just as before this shows that $B = 0$, which now implies that $E \cong A$ has all stable factors isomorphic to S. Therefore, in the Grothendieck group we have $[E] = n[S]$ for some n. If E is skyscraper sheaf applying degree to both sides shows n must be 1. If E is a line bundle, applying rank to both sides shows $n = 1$. In either case we have that $E \cong S$ is stable. □

And now we can prove that the $\widetilde{GL^+}(2, \mathbb{R})$ action on numerical stability conditions is free and transitive:

Proof of Theorem. Let $\sigma = (Z, \mathcal{P})$ be an arbitrary numerical stability condition on a curve X of positive genus. Since $\mathrm{Hom}(L, k_x) \neq 0$ for any line bundle L and skyscraper sheaf k_x, and they are both stable, we get that $\phi(L) < \phi(k_x)$. By Serre duality, $\mathrm{Ext}^1(k_x, L) \cong \mathrm{Hom}(L, k_X \otimes \omega_x)^\vee \neq 0$, so $\phi(k_x) - 1 < \phi(L)$.

Notice that so far, we haven't really used the fact that σ is numerical stability condition. Recall from our discussion of the numerical Grothendieck group that degree and rank give an isomorphism between $\mathcal{N}(X)$ and $\mathbb{Z}^2$. Under that isomorphism, any skyscraper sheaf is sent to $(0, 1)$ and a line bundle is sent to a point of the form $(1, d)$. This already implies that all skyscraper sheaves have phases that differ by even numbers (they have the same central charge), but in fact, since the above inequalities imply that for two different skyscrapers but a single line bundle we have $\phi(k_x) - 1 < \phi(L) < \phi(k_y)$ and $\phi(k_y) - 1 < \phi(L) < \phi(k_x)$, we see that we must have $\phi(k_x) = \phi(k_y)$. Similarly, the phase of any line bundle depends only on its degree (an improvement over "its phase mod 2 depends only on the degree").

I claim the inequalities for the phase we showed above already imply that the is some matrix $M \in GL^+(2, \mathbb{R})$ such that $M^{-1} \circ Z$ will equal the slope-stability central charge, $-\deg + i\,\mathrm{rank}$. Indeed, the inequalities show that a *unique* such matrix can be chosen so that these two central charges agree on all skyscrapers and one specific line bundle. Then one can easily show that they agree on all line bundles by using appropriate short exact sequences connecting different line bundles. Finally, since line bundles and skyscraper sheaves generate the Grothendieck group, this means the central charges agree everywhere.

After picking any element of $\widetilde{GL^+}(2, \mathbb{R})$ covering the matrix M we reduce to the case where Z is the usual slope-stability central charge, but where the phases of skyscraper sheaves might be off from the standard one by a fixed even integer. That can also be corrected by the action of an element of $\widetilde{GL^+}(2, \mathbb{R})$. After doing that we get a t-structure whose heart contains all skyscraper sheaves and all line bundles, and thus contains all coherent sheaves. This forces this new heart to be exactly $\mathrm{Coh}(X)$: given two hearts $\mathcal{A}$ and $\mathcal{B}$ of bounded t-structures, if $\mathcal{A} \subseteq \mathcal{B}$ we must have equality. This is easy to see by taking an object B of $\mathcal{B}$ and looking at the filtration of it whose quotients lie in shifts of $\mathcal{A}$: each shift of $\mathcal{A}$ is contained in the corresponding shift of $\mathcal{B}$, which shows the filtration must consist of just one object, namely B, which is therefore in $\mathcal{A}$. $\square$

References

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