280x (22/10)

References for this talk

1. "Vector bundles on elliptic curves", Atiyah.
2. Last 2 pages of Bridgeland paper.
3. "Some examples of stability conditions...", Macri.

Our purpose is: to describe the space of numerical stability conditions on an elliptic curve, or more generally a smooth proj curve of genus $g > 0$. Also discuss what we miss by focusing on "numerical".

factor via the numerical Grothendieck group

First, let's discuss v.b. on elliptic curve E.

Probably should say first that for any sheaf $\mathcal{F} \in \mathrm{Coh}(E)$

Defn. Let M be a module over a ring A. An element $m \in M$ is <u>torsion</u> if for some $a \in A$, a not a zero divisor $am = 0$. Can define $M_{tor} \subseteq M$ sub module of torsion elements. The global version is a presheaf. We can sheafify $\Rightarrow$ a torsion sheaf $\mathcal{F}_{tor} \subseteq \mathcal{F}$.

<u>Claim.</u> For any elliptic curve E, $\mathcal{F} \in \mathrm{Coh}(E)$

$$\mathcal{F} \cong \mathcal{F}_{tor} \oplus \mathcal{F}/\mathcal{F}_{tor}$$

So we need to classify indecomposable vector bundles and indecomposable torsion sheafs.

Let me remind you what rank of a vector bundle means

Defn. If $\mathcal{L}$ is a line bundle on E, $\mathcal{L}$ correspond a sum of ~~[illegible]~~ points i.e. $\mathcal{L} \cong \mathcal{O}(-D)$ with D divisor. In the complex picture this is a space of meromorphic functions with allowed poles and required zeros. We can write $D = \sum a_p \cdot [p]$. Then define

$$\deg \mathcal{O}(-D) = -\sum a_p$$

This is the same ~~Think of this~~ as taking a rational section of $\mathcal{L}$ and counting poles and zeros

<u>Reminder:</u> Karishma showed

$$\chi(\mathcal{F}) = \sum (-1)^i H^i(\mathcal{F})$$

$$\deg \mathcal{L} = \chi(\mathcal{L}) - \underbrace{\chi(\mathcal{O}_X)}_{1-\text{genus}}$$

In the case of a curve, and $\mathcal{F}$ a v.b. (maybe also coherent sheaf?)

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) \underset{\text{Serre duality}}{=} h^0(\mathcal{F}) - h^0(\mathcal{F}^\vee \otimes \omega_E)$$

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Defn. for ~~vector bundle~~ coherent sheaf $\mathcal{F}$ we define

$$\deg \mathcal{F} = \chi(\mathcal{F}) - \mathrm{rk}\, \mathcal{F} \cdot \chi(\mathcal{O}_X)$$

<u>It is additive for SES</u> (built from Euler characteristic and rank). <u>For vector bundles</u>: $\deg \mathcal{F} = \deg(\det \mathcal{F})$.

Defn. $\mathrm{rank}(\mathcal{F}) = \dim_{\mathcal{O}_{X,\eta}} \mathcal{F}_\eta$ where η is the generic point.

Example. Think $C = \mathbb{A}^1$, the trivial line bundle $\widetilde{k[t]}$ and the map

$$0 \to \widetilde{k[t]} \xrightarrow{\times t} \widetilde{k[t]} \longrightarrow \widetilde{k[t]/(t)} \to 0$$

the dimension jumps at $t=0$

so that's why we take dimension at the generic point.

<u>Thm</u> (Atiyah) For every rank r, degree d

$$V_{r,d} = \left\{ \begin{array}{l} \text{indecomposable v.b. of} \\ \text{rank } r \text{ and deg } d \end{array} \right\} \cong E$$

This generalizes the classical statement about the Jacobian of elliptic curve, in addition There is a correspondence

$$\begin{tikzcd} V_{r,d} \arrow[r, "\det"] \arrow[d] & V_{1,d} \arrow[d] \\ E \arrow[r, "\gcd(r,d)"] & E \end{tikzcd}$$

$$x \longmapsto \gcd(r,d)\cdot x$$

More concretely, $E_{r,d} \in V_{r,d}$ then

$$V_{r,d} \cong \{ E_{r,d} \otimes \mathcal{L} \mid \mathcal{L} \text{ line bundle of degree } 0 \}$$

this is almost unique, because

$$E_{r,d} \otimes \mathcal{L} \cong E_{r,d} \Rightarrow \mathcal{L}^r \cong \mathcal{O}_X.$$

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This is very reasonable - if $\mathcal{L}^{\otimes r} \cong \mathcal{O}_X$

$$\mathcal{E} = \mathcal{O}_X \oplus \mathcal{L} \otimes \_\_\_ \otimes \mathcal{L}^{\otimes(n-1)}$$

Then $\mathcal{E} \otimes \mathcal{L} \cong \mathcal{E}$.

A consequence of this, is that looking at Grothendick group of an elliptic curve

$$K_0(D^b(\mathrm{coh}\, E)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Pic}^0(E)$$

Define the numerical Grothendieck group

$$N(D^b(\mathrm{coh}\, E)) = \mathbb{Z} \oplus \mathbb{Z}$$

i.e. rememebering only rank and degree.

Recall we have Mukai pairing

$$\langle \mathcal{E}, \mathcal{F} \rangle = \sum (-1)^i \dim \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$$

$$= \sum (-1)^i \dim \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$$

morphism in derived category

It is bilinear on K_0.

Can look at left/right radicals or kernels

left kernel $:= \{\mathcal{E} \mid \langle \mathcal{E}, \mathcal{F} \rangle = 0 \ \forall \mathcal{F}\}$

right kernel $:= \{\mathcal{E} \mid \langle \mathcal{F}, \mathcal{E} \rangle = 0 \ \forall \mathcal{F}\}$

Serre duality gives a relation. The Serre functor $S(\mathcal{E}) = \mathcal{E} \otimes \omega_X[\dim X]$ implies $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(SB, A)^\vee$ so the pairing between $\mathcal{E}$ and $\mathcal{F}$

$$\langle \mathcal{E}, \mathcal{F} \rangle = \langle \mathcal{F}, S(\mathcal{E}) \rangle$$

So the right kernel is the image of S applied to the left kernel. For an elliptic curve this is really easy - the rank is 1 and

$$\langle \mathcal{E}, \mathcal{F} \rangle = - \langle \mathcal{F}, \mathcal{E} \rangle$$

In particular, the left and right kernels are the same. So $N = K_0 / \ker\langle \, , \rangle$.

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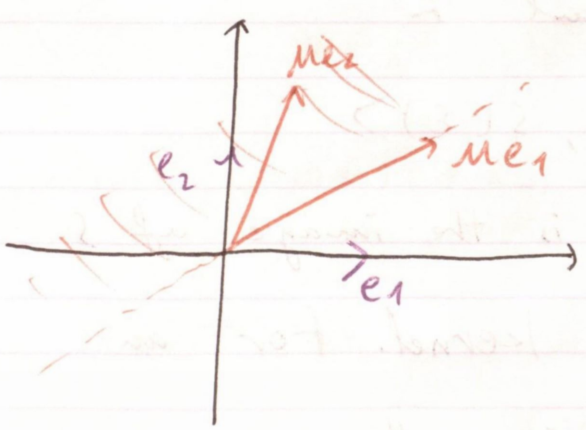
Now we want to describe the action of $\widetilde{GL^+}(2,\mathbb{R})$ on $Stab(X)$. Let's describe this universal cover first.

Homotopically, by Gram-Schmidt there is a deformation retract into $SO(2,\mathbb{R})$. So there is a homotopy equivalence

$$GL^+(2,\mathbb{R}) \simeq SO(2,\mathbb{R}) \cong S^1$$



Me_1 to Me_2 in upper half plane above Me_1.

So if you want to describe lift to the universal cover, you just need to keep track of how many times you around the origin. So

$$\widetilde{GL}^+(2,\mathbb{R}) = \{ (M,f) \mid M \in GL^+(2,\mathbb{R})$$

$f:\mathbb{R}\to\mathbb{R}$, increasing, $f(\phi)+1 = f(\phi+1)$,

$Me^{i\theta} \in \mathbb{R}_+ e^{if(\theta)} \}$

Get a path in $\mathbb{R}^2 \setminus \{(0,0)\}$ from $Me^{i\theta}$.

Thus $\dfrac{Me^{i\theta}}{\|Me^{i\pi\theta}\|}$ gives diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[d] & \mathbb{R} \arrow[d] \\
S^1 \arrow[r, "\frac{Me^{i\pi\theta}}{|Me^{i\pi\theta}|}"'] & S^1
\end{tikzcd}$$

$\widetilde{GL}^+(2,\mathbb{R})$ acts on space of stability conditions

$$(M,f)\cdot(Z,P) = (M^{-1}\circ Z,\ P\circ f)$$

by a right action. There is also left action of $\mathrm{Aut}(D^b \mathrm{Coh}\, X)$ on $\mathrm{Stab}(X)$: given a self equiv Φ we can define

$$\Phi\circ(Z,P) = (Z\circ\Phi^{-1},\ \Phi P)$$

use Φ before central charge

image of a slice under P.

$\bar{\Phi}$ induced by Φ on $K_0(D^b \operatorname{coh} X)$. Basically, they are the obvious group actions.

<u>Q:</u> Can we use Qwashu to make more explicit?

<u>A</u> Probably yes.

Next, we'll explain that if X is smooth proj of genus >0. Then the numerical stability condition which we'll just denote $\mathrm{Stab}(X)$ are homeomorphic to $\widetilde{GL}^+(2,\mathbb{R})\cdot\sigma$ where $\sigma(E) = -\deg(E) + i\,\mathrm{rk}(E)$.

So the action is free and transitive and all stability condition are image of the standard one.

One thing we'll need and forget ~~to~~ to mention is that $D^b \mathrm{Coh} X$ is of homological dimension 1. It's a consequence of Hilbert syzygy thm. In practice, what we'll use is that

$$\mathrm{Ext}^{\geq 2}(E, F) = 0$$

for $E, F \in \mathrm{Coh}(X)$.

By Serre duality

$$\mathrm{Ext}^2(E, F) = \mathrm{Ext}^{-2}(F, E \otimes \omega_X) = 0$$

so this is obvious. One thing that has to happen in such a category is that every

$E \in D^b \mathrm{Coh}(X)$ we have $E \cong \bigoplus \mathcal{H}^i(E)[-i]$

cohomology sheaves

in the derived category

i.e. can always replace a complex with an equivalent one without differentials.

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$$E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \longrightarrow \cdots$$

two ways to truncate.

τ_{i+1} truncate at $i+1$

$$\begin{tikzcd} \cdots \arrow[r] & E^{i-1} \arrow[r] \arrow[d] & E^i \arrow[r] \arrow[d] & E^{i+1} \arrow[r] \arrow[d] & \operatorname{im} d^{i+1} \arrow[r] & 0 \\ \cdots \arrow[r] & E^{i-1} \arrow[r] & E^i \arrow[r] & \ker d^{i+1} \arrow[r] & 0 \end{tikzcd}$$

Another way

This are two equiv truncations. If we truncate one step before

$$\begin{tikzcd} \tau_i & E^{i-1} \arrow[r] \arrow[d] & E^i \arrow[r] \arrow[d] & \operatorname{im} d^i \arrow[d] \\ \tau_{i+1} & E^{i-1} \arrow[r] & E^i \arrow[r] & \ker d^{i+1} \arrow[d] \\ & 0 \arrow[r] & 0 \arrow[r] & H^{i+1}(E) \arrow[r] & 0 \end{tikzcd}$$

and exact triangle

$$\tau_i \mathcal{E} \to \tau_{i+1} \mathcal{E} \to \mathcal{H}^{i+1}(\mathcal{E})[i-1]$$

We use induction and this. We can assume $\tau_i \mathcal{E} \cong \bigoplus_{j \le i} \mathcal{H}^j(\mathcal{E})[-j]$, $\tau_{i+1} \cong \mathcal{E}$

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to show the induction step enough to show extension is trivial. So we look at

$$Hom(\mathcal{H}^{i+1}(\mathcal{E})[-i-1], \bigoplus_{j\le i} \mathcal{H}^j(\mathcal{E})[-j+1])$$

$Ext^1(-,-) = Hom(-,-[1])$

This is

$$\bigoplus_{j\le i} Hom(\mathcal{H}^{i+1}(\mathcal{E}), \mathcal{H}^j(\mathcal{E})[\underbrace{2+i-j}_{\ge 2}]) = 0$$

$\square$

(1) Main lemma we'll need. • If (Z, P) is a numerical stability condition on $D^b Coh(X)$, all k_x and all line bundles are σ-stable.

To prove this need lemma of Gorodentsev, Kuleshov, Rudakov:

Lemma. Given an exact triangle

(2)

$$A \to E \to B$$

$E \in Coh(X)$ and $Ext^{\le 0}(A,B) = 0$ then $A, B \in Coh(X)$ as well.

This is just some triangulated ~~ct~~ theory.

(3) ~~[illegible]~~ Mild reform that does not require genus>0

$A \to E \to B$, $E \in Coh(X)$

and $Ext^{<0}(A, B) = 0 \Rightarrow$

$A \cong A_0 \oplus A_1$

$B \cong B_0 \oplus B_{-1}$

If you know genus>0 and $Ext^0(A,B) = 0$

then we can improve this to deduce they are in the heart. So the proof of this lemma has 2 steps. (2)+(3)

Start with (3). $A \cong \oplus A_i[-i]$

$B \cong \oplus B_j[-j]$

~~[illegible]~~ by looking at LES, get sequence

$0 \to B_{-1} \to A_0 \to E \to B_0 \to A_1 \to 0$

and $A_i \cong B_{i-1}$ for $i \neq 0, 1$

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Well, if $A_i \neq 0$ for $i \neq 0,1$ then we'd have:

$$id \in Hom(A_i, B_{i-1}) = Ext^{-1}(A_i[-i], B_{i-1}[-i+1])$$

$$\Rightarrow Ext^{-1}(A,B) \neq 0$$

and that's a contradiction.