STABILITY CONDITIONS ON $\mathbb{P}^1$ AND WALL-CROSSING

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Abstract. We sketch a proof that the stability manifold of the bounded derived category of coherent sheaves on $\mathbb{P}^1$, $\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$, is isomorphic to $\mathbb{C}^2$ as a complex manifold. We also discuss the notion of wall-crossing for three hearts for the t-structure on $\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$, namely $\mathrm{Coh}(\mathbb{P}^1)$, $\mathrm{Rep}(\bullet \rightrightarrows \bullet)$ and $\mathrm{Rep}(\bullet\ \bullet)$.

1. Preliminaries

In [Bri07], Bridgeland introduced the notion of a stability condition $(Z,\mathscr{P})$ on a triangulated category $\mathscr{D}$. This is an abstraction of the properties of μ-stability for sheaves on projective varieties. Here the *central charge* $Z : K(\mathscr{D}) \to \mathbb{C}$ is a group homomorphism from the Grothendieck group of $\mathscr{D}$ to the complex numbers and $\mathscr{P}$ is a *slicing* of $\mathscr{D}$, that is, there are full additive subcategories $\mathscr{P}(\phi) \subset \mathscr{D}$ for each $\phi \in \mathbb{R}$, satisfying the following compability conditions:

(a) if $E \in \mathscr{P}(\phi)$, then $Z(E) = m(E)e^{i\pi\phi}$ for some $m(E) \in \mathbb{R}_{>0}$,
(b) for all $\phi \in \mathbb{R}$, $\mathscr{P}(\phi+1) = \mathscr{P}(\phi)[1]$,
(c) if $\phi_1 > \phi_2$ and $A_j \in \mathscr{P}(\phi_j)$, then $\mathrm{Hom}_{\mathscr{D}}(A_1,A_2) = 0$,
(d) (Harder-Narasimhan property) for each nonzero object $E \in \mathscr{D}$, there are a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathscr{P}(\phi_j)$ for all j.

Bridgeland showed that the set of stability conditions $\mathrm{Stab}(\mathscr{D})$ on a fixed category $\mathscr{D}$ has a natural topology, and that $\mathrm{Stab}(\mathscr{D})$ is in fact a manifold under this topology. Macrì [Mac04] showed that for a smooth projective curve C over $\mathbb{C}$ of positive genus, $\mathrm{Stab}(\mathrm{D^b}\,\mathrm{Coh}(C)) \cong \widetilde{GL^+}(2,\mathbb{R})$, the universal cover of the group of rank two matrices with positive determinant.

The main theorem of this talk is the following:

Theorem 1 (Okada [Oka04])**.** *The stability manifold for the bounded derived category of coherent sheaves on $\mathbb{P}^1$, $\mathrm{Stab}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1))$, is isomorphic to $\mathbb{C}^2$ as a complex manifold.*

The strategy is to show that the quotient of $\mathrm{Stab}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1))$ for a certain action of $\mathbb{C}\times\mathbb{Z}$ is isomorphic to $\mathbb{C}^*$. The $\mathbb{C}$-action is defined as follows:

Definition 2. Let $(Z,\mathscr{P})$ be a stability condition and $z = x+iy \in \mathbb{C}$. Then $z*(Z,\mathscr{P})$ is defined by $z*Z = e^zZ$ and $(z*\mathscr{P})(\phi) = \mathscr{P}(\phi - y/\pi)$.

Remark 3. The real and imaginary parts of z act as rescaling and rotation respectively. Rotation affects the heart of a triangulated category but preserves semistable objects (defined below), while wall-crossing, which we will discuss later, preserves the heart but affects semistable objects.

Definition 4. A *stability function* on an abelian category $\mathscr{A}$ is a group homomorphism $Z : K(\mathscr{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathscr{A}$, the complex number $Z(E)$ lies in

$$H := \{re^{i\pi\phi} : r > 0 \text{ and } 0 < \phi \leq 1\} = \mathbb{H} \cup \mathbb{R}_{<0} \subset \mathbb{C},$$

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where $\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$ is the complex upper half-plane.

Given a stability function $Z : K(\mathscr{A}) \to \mathbb{C}$, the *phase* of an object $0 \neq E \in \mathscr{A}$ is defined to be

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

An object $0 \neq E \in \mathscr{A}$ is said to be *semistable* if every subobject $0 \neq A \subset E$ satisfies $\phi(A) \leq \phi(E)$.

Example 5. Recall that the heart of a triangulated category is always an abelian category. A stability function on the heart $\mathrm{Coh}(\mathbb{P}^1)$ of the triangulated category $\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$ is given by a choice of $Z(k_x) \in \mathbb{R}_{<0}$ and $Z(\mathscr{O}) \in \mathbb{H}$, where k_x is a skyscraper sheaf. We illustrate the standard stability function

$$\begin{aligned} Z_{\mathrm{std}} : K(\mathrm{Coh}(\mathbb{P}^1)) &\to \mathbb{C}, \\ E &\mapsto -\deg E + i \operatorname{rk} E, \end{aligned}$$

which is given by $Z(k_x) = -1$ and $Z(\mathscr{O}) = i$, below.

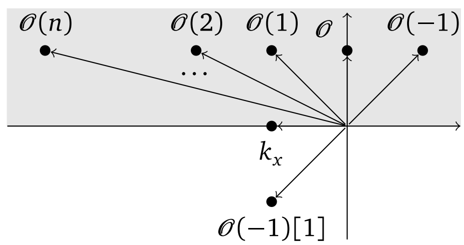


FIGURE 1. Standard stability function $Z_{\mathrm{std}}(E) = -\deg E + i \operatorname{rk} E$ on $\mathrm{Coh}(\mathbb{P}^1)$. Every object in $\mathrm{Coh}(\mathbb{P}^1)$ has image in H (shaded gray).

By a lemma of Bridgeland, since Z satisfies the Harder-Narasimhan property, it extends to a stability condition $(Z, \mathscr{P})$ on $\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$. Rotation by $z = -i\phi(\mathscr{O}(-1))\pi$ gives a stability condition $z * (Z, \mathscr{P}) = (\overline{Z}, \overline{\mathscr{P}})$ such that the heart $\overline{\mathscr{P}}((0, 1])$ is equivalent to $\mathrm{Rep}(\bullet \rightrightarrows \bullet)$. We see this as follows.

A theorem of Beilinson says that there is an equivalence of triangulated categories

$$\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1) \to \mathrm{D^b}\,\mathrm{Rep}(\bullet \rightrightarrows \bullet).$$

Recall from class that

$$\mathrm{Rep}(1\bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bullet 2)$$

is equivalent to the category of left modules on the path algebra $k(\bullet \rightrightarrows \bullet)$ of the quiver $\bullet \rightrightarrows \bullet$ (i.e. the free (non-unital) associative algebra generated by the vertices and arrows of the quiver subject to the expected rules of composition), and the quiver representation corresponding to $k(\bullet \rightrightarrows \bullet)$ as a $k(\bullet \rightrightarrows \bullet)$-module is

$$k\langle e_1 \rangle \overset{f_\alpha}{\underset{f_\beta}{\rightrightarrows}} k\langle e_2 \rangle \oplus k\langle \alpha \rangle \oplus k\langle \beta \rangle.$$

Note that elements of the algebra act on elements of the module by postcomposition. Writing this out in tabular format, we have

$k(\bullet \rightrightarrows \bullet)$ as algebra \ $k(\bullet \rightrightarrows \bullet)$ as module	e_1	e_2	f_α	f_β
e_1	e_1	0	0	0
e_2	0	e_2	f_α	f_β
f_α	f_α	0	0	0
f_β	f_β	0	0	0

TABLE 1. Action of $k(\bullet \rightrightarrows \bullet)$ on itself

Beilinson's equivalence is given by

$$\begin{aligned} \mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1) &\to \mathrm{D^b}\,\mathrm{Rep}(\bullet \rightrightarrows \bullet) \simeq \mathrm{D^b}(k\langle e_1\rangle \overset{f_\alpha}{\underset{f_\beta}{\rightrightarrows}} k\langle e_2\rangle \oplus k\langle\alpha\rangle \oplus k\langle\beta\rangle) \\ E &\mapsto \mathrm{Ext}^\bullet(\mathscr{O} \oplus \mathscr{O}(1), E). \end{aligned}$$

Let us do a few calculations. We have

$$\mathrm{Hom}(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}) = \mathrm{Hom}(\mathscr{O}, \mathscr{O}) \oplus \mathrm{Hom}(\mathscr{O}(1), \mathscr{O}) = k\langle \mathrm{id}_{\mathscr{O}}\rangle,$$

$$\mathrm{Ext}^1(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}) \cong \mathrm{Hom}(\mathscr{O}, (\mathscr{O} \oplus \mathscr{O}(1)) \otimes \mathscr{O}(-2)) = \mathrm{Hom}(\mathscr{O}, \mathscr{O}(-1) \oplus \mathscr{O}(-2)) = 0$$

by Serre duality, since the dualizing sheaf of $\mathbb{P}^1$ is $\mathscr{O}(-2)$, and we know that

$$\mathrm{Ext}^i(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}) = 0 \quad \text{for} \quad i \neq 0, 1$$

because the abelian category of coherent sheaves on a smooth projective variety X of dimension n has homological dimension at most n (i.e. $\mathrm{Ext}^i(\mathscr{F}, \mathscr{G}) = 0$ for $i > n$ and any two coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on X).

We also have

$$\mathrm{Hom}(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}(-1)[1]) = \mathrm{Ext}^1(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}(-1)) \cong \mathrm{Hom}(\mathscr{O}(-1), (\mathscr{O} \oplus \mathscr{O}(1)) \otimes \mathscr{O}(-2)) = k\langle \mathrm{id}_{\mathscr{O}(-1)}\rangle$$

by Serre duality and using $\mathrm{Hom}(\mathscr{O}(m)[a], \mathscr{O}(n)[b]) = \mathrm{Ext}^{b-a}(\mathscr{O}(m), \mathscr{O}(n))$,

$$\mathrm{Ext}^1(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}(-1)[1]) = \mathrm{Hom}(\mathscr{O}(-1)[1], (\mathscr{O} \oplus \mathscr{O}(-1)) \otimes \mathscr{O}(-2)) = \mathrm{Ext}^{-1}(\mathscr{O}(-1), \mathscr{O}(-2) \oplus \mathscr{O}(-3)) = 0.$$

Again, using the result on the homological dimension of $\mathrm{Coh}(\mathbb{P}^1)$, we conclude that $\mathrm{Ext}^i(\mathscr{O} \oplus \mathscr{O}(1), \mathscr{O}(-1)[1]) = 0$ for $i \neq 0$.

There is a two-dimensional space of maps $\mathscr{O} \to \mathscr{O}(1)$. For a suitable choice of coordinates x_0, x_1 on $\mathbb{P}^1$, let us abuse notation and call the corresponding generators of this space of maps x_0 and x_1 also. Similarly, there is an action of

$$\mathrm{End}(\mathscr{O} \oplus \mathscr{O}(1)) = \mathrm{End}(\mathscr{O}) \oplus \mathrm{End}(\mathscr{O}(1)) \oplus \mathrm{Hom}(\mathscr{O}, \mathscr{O}(1)) \oplus \mathrm{Hom}(\mathscr{O}(1), \mathscr{O}) = k\langle\mathscr{O}\rangle \oplus k\langle\mathscr{O}(1)\rangle \oplus k\langle x_0\rangle \oplus k\langle x_1\rangle$$

on $\mathrm{Hom}(\mathscr{O} \oplus \mathscr{O}(1), E)$ by precomposition:

$\mathrm{End}(\mathscr{O} \oplus \mathscr{O}(1))$ as module / $\mathrm{End}(\mathscr{O} \oplus \mathscr{O}(1))$ as algebra	$\mathrm{id}_{\mathscr{O}(1)}$	$\mathrm{id}_{\mathscr{O}}$	x_0	x_1
$\mathrm{id}_{\mathscr{O}(1)}$	$\mathrm{id}_{\mathscr{O}(1)}$	0	0	0
$\mathrm{id}_{\mathscr{O}}$	0	$\mathrm{id}_{\mathscr{O}}$	x_0	x_1
x_0	x_0	0	0	0
x_1	x_1	0	0	0

TABLE 2. Action of $\mathrm{End}(\mathscr{O} \oplus \mathscr{O}(1))$ on itself

Comparing Table 2 with Table 1, we conclude that $\mathscr{O}$ corresponds to the subquiver $0 \rightrightarrows k$ and $\mathscr{O}(1)$ corresponds to $k \rightrightarrows 0$.

Consider the result of rotating the heart in Figure 1 by $z = -i\phi(\mathscr{O}(-1))\pi$. We obtain the heart in Figure 2a, which we showed above corresponds to the heart in Figure 2b. Thus, using the equivalence above, we see that $\overline{\mathscr{P}}(0, 1]$ is generated by $\mathscr{O}$ and $\mathscr{O}(-1)[1]$.

(A) Z_{std} on $\mathrm{Coh}(\mathbb{P}^1)$ rotated by $z = -i\phi(\mathscr{O}(-1)[1])\pi$ (B) Stability function on $\mathrm{Rep}(\bullet \rightrightarrows \bullet)$

FIGURE 2. Equivalence of hearts

We can express this using the commutative diagram below:

$$\begin{tikzcd} K(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)) \arrow[r, "\simeq"] \arrow[d] & K(\mathrm{D^b}\,\mathrm{Rep}(\bullet \rightrightarrows \bullet)) \arrow[d] \\ \mathbb{Z}^2 \arrow[r, "A"] & \mathbb{Z}^2 \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, mapsto] \arrow[d, mapsto] & k^a \rightrightarrows k^b \arrow[d, mapsto] \\ (-\deg E, \operatorname{rk} E) \arrow[r, mapsto] & (a,b) \end{tikzcd}$$

From the diagrams

$$\begin{tikzcd} \mathscr{O} \arrow[r, mapsto] \arrow[d, mapsto] & 0 \rightrightarrows k \arrow[d, mapsto] \\ (0,1) \arrow[r, mapsto] & (0,1) \end{tikzcd} \quad \begin{tikzcd} \mathscr{O}(-1)[1] \arrow[r, mapsto] \arrow[d, mapsto] & k \rightrightarrows 0 \arrow[d, mapsto] \\ (-1,-1) \arrow[r, mapsto] & (1,0) \end{tikzcd} \quad \begin{tikzcd} k_x \arrow[r, mapsto] \arrow[d, mapsto] & k \rightrightarrows k \arrow[d, mapsto] \\ (-1,0) \arrow[r, mapsto] & (1,1) \end{tikzcd}$$

we see that the matrix of the linear transformation is

$$A = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}.$$

2. Wall Crossing

We now consider what happens to the semistable objects when we fix $\overline{\mathscr{P}}((0,1])$ and vary the stability function $\widetilde{Z}$ on it. From the above example, we note that $\widetilde{Z}$ is determined by a choice of values $z_0 = \widetilde{Z}(\mathscr{O})$, $z_{-1} = \widetilde{Z}(\mathscr{O}(-1)[1]) \in H$.

There are three cases, whose loci partition $\mathrm{Stab}_{\mathrm{Rep}(\bullet\rightrightarrows\bullet)}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1))$, the locus of stability conditions on $\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$ with heart $\overline{\mathscr{P}}(0,1] \simeq \mathrm{Rep}(\bullet \rightrightarrows \bullet)$.

- $\phi(z_0) < \phi(z_{-1})$. This is the case we considered above, where $\widetilde{Z}$ is equivalent to the standard stability function $Z_{\mathrm{std}}(E) = -\deg(E) + i\,\mathrm{rk}(E)$ up to reparametrization of $\mathbb{C}$ by an element in $GL^+(2,\mathbb{R})$ (and accordingly adjusting the phases of objects).
- $\phi(z_0) = \phi(z_{-1})$. Then any non-zero object in $\overline{\mathscr{P}}((0,1])$ is semistable since all non-zero objects have the same slope.

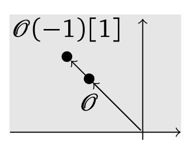
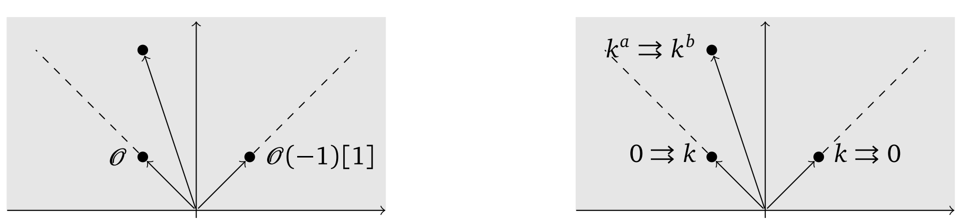


Figure 3. Case when the generators $z_0 = \widetilde{Z}(\mathscr{O})$ and $z_{-1} = \widetilde{Z}(\mathscr{O}(-1)[1])$ of $\operatorname{im} Z$ have the same slope.

- $\phi(z_0) > \phi(z_{-1})$. Then any quiver $k^a \rightrightarrows k^b$ not of the form $k^a \rightrightarrows 0$ has a subobject $0 \rightrightarrows k$ of larger phase unless the original quiver had the form $0 \rightrightarrows k^b$, so the multiples of $\mathscr{O}$ and $\mathscr{O}(-1)[1]$ are the only semistable objects in $\overline{\mathscr{P}}((0,1])$.



(A) On $\mathscr{P}(0,1]$ (B) On $\mathrm{Rep}(\bullet \rightrightarrows \bullet)$

Figure 4. The multiples of $\mathscr{O}$ and $\mathscr{O}(-1)[1]$ are the only semistable objects when $z_0 = \widetilde{Z}(\mathscr{O})$ has larger phase than $z_{-1} = \widetilde{Z}(\mathscr{O}(-1)[1])$.

Thus we see that the semistable objects change when as we pass through the locus $\phi(z_0) = \phi(z_{-1})$. In particular, the skyscraper sheaf k_x is an example of an object for which the property of being semistable changes as we pass through the locus $\phi(z_0) = \phi(z_{-1})$. This leads us to the following definition, of which the locus $\{(Z,\mathscr{P}) \in \mathrm{Stab}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)) \mid \phi(z_0) = \phi(z_{-1})\}$ is an example.

Definition 6. A *wall* is a codimension one submanifold of a stability manifold such that as one varies a stability condition, a semistable object can only become non-semistable if one crosses a wall.

Aside. Suppose that we are in the last case $\phi(z_0) > \phi(z_{-1})$ and we deform Z such that z_{-1} leaves the upper half plane by crossing the positive real line. In this case, the semistable objects do not change; however, we obtain a new heart $\mathscr{A}' = \widetilde{\mathscr{P}}((0,1])$ generated by the stable objects $\mathscr{O}$ and $\mathscr{O}(-1)[2]$.

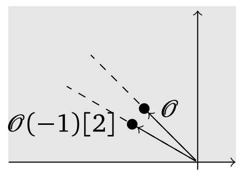


FIGURE 5. Case when the generators $z_0 = \widetilde{Z}(\mathscr{O})$ and $z_{-1} = \widetilde{Z}(\mathscr{O}(-1)[1])$ of $\mathrm{im}\,Z$ have the same slope.

Using $\mathrm{Hom}(\mathscr{O}(m)[a], \mathscr{O}(n)[b]) = \mathrm{Ext}^{b-a}(\mathscr{O}(m), \mathscr{O}(n))$ and the result on the homological dimension of $\mathrm{Coh}(\mathbb{P}^1)$ a last time, we see that there are no morphisms or extensions between these two objects, so $\mathscr{A}'$ is isomorphic to the category of pairs of vector spaces and thus it is isomorphic to $\mathrm{Rep}(\bullet\ \bullet)$. This is an example of a heart $\mathscr{A}'$ of $\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$ such that $\mathrm{D^b}(\mathscr{A}') \neq \mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1)$.

The following is a generalization of our observations above to any heart.

Lemma 7. *Up to the action of* $\mathrm{Aut}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1))$*, for any stability condition on* $\mathrm{Stab}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1))$ *(i.e. not necessarily one with heart $\mathscr{P}(0,1]$), there exists some $p > 0$ such that $\mathscr{O}(-1)[p]$ and $\mathscr{O}$ are semistable and $\phi(\mathscr{O}(-1)[p])$, $\phi(\mathscr{O}) \in (r, r+1]$ (i.e. $\phi(\mathscr{O}) - \phi(\mathscr{O}(-1)[1]) > -1$) for some $r \in \mathbb{R}$.*

If $\phi(Z(\mathscr{O}(-1)[1])) < \phi(Z(\mathscr{O})$, the multiples of the shifts of $\mathscr{O}(-1)$ and $\mathscr{O}$ are the only semistable objects. If $\phi(Z(\mathscr{O}(-1)[1])) > \phi(Z(\mathscr{O})$, then all line bundles and torsion sheaves are semistable.

In fact all cases in the above lemma exist:

Proposition 8. *For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha - \beta > -1$ and any m_α, $m_\beta \in \mathbb{R}_{>0}$, there exists a unique stability condition $(Z,\mathscr{P})$ such that $\phi(\mathscr{O}(-1)[1]) = \beta$ and $\phi(\mathscr{O}) = \alpha$, and $z_0 = m_\alpha e^{i\pi\alpha}$, $z_{-1} = m_\beta e^{i\pi\beta}$. Moreover, we have the following cases:*

- *if $\alpha > \beta$, then for any $r \in \mathbb{R}$, there exist $p, q \in \mathbb{Z}$ such that $\mathscr{P}((r-1, r]) = \langle \mathscr{O}(-1)[p+1], \mathscr{O}[q] \rangle$ and $p - q \in (\alpha - \beta - 1, \alpha - \beta + 1)$;*
- *if $\alpha = \beta$, then for any r, there exists j such that $\langle \mathscr{O}(-1)[j+1], \mathscr{O}(j) \rangle = \mathscr{P}((r-1, r])$;*
- *if $\alpha < \beta$, then for any r, either there exist $i, j \in \mathbb{Z}$ such that $\mathscr{P}((r-1, r]) = \langle \mathscr{O}(i-1)[j+1], \mathscr{O}(i)[j] \rangle$ and $\phi(\mathscr{O}(i-2)[j+1]) > r \geq \phi(\mathscr{O}(i-1)[j+1])$, or there exists $j \in \mathbb{Z}$ such that $\mathscr{P}((r-1, r]) = \mathrm{Coh}\,\mathbb{P}^1[j]$ and $r = \phi(k_x[j])$.*

Corollary 9. *The hearts on which there exists a stability function satisfying the Harder-Narasimhan property are* $\mathrm{Coh}\,\mathbb{P}^1[j]$ *and $\langle \mathscr{O}(i-1)[p+j], \mathscr{O}(i)[j] \rangle$ for all $i, j \in \mathbb{Z}$ and $p > 0$.*

3. PROOF OF THE MAIN THEOREM

We now return to the $\mathbb{C}$-action on $\mathrm{Stab}(\mathrm{D^b}\,\mathrm{Coh}(\mathbb{P}^1))$. As mentioned in the previous talk, $\widetilde{GL^+(2,\mathbb{R})}$ acts on $\mathrm{Stab}(\mathscr{D})$ for any triangulated category $\mathscr{D}$ as follows. Regard $GL^+(2,\mathbb{R})$ as the set of pairs (T, f) where $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi+1) = f(\phi) + 1$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving linear isomorphism such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$ are the same. Given a stability condition $(Z,\mathscr{P}) \in \mathrm{Stab}(\mathscr{D})$ and a pair $(T, f) \in \widetilde{GL^+(2,\mathbb{R})}$, define a new stability condition $(Z', \mathscr{P}')$ by setting $Z' = T^{-1} \circ Z$ and $\mathscr{P}'(\phi) = \mathscr{P}(f(\phi))$. Note that the semistable objects do not change, but the phases have been relabeled.

Proposition 10. *The $\mathbb{C}$-action is holomorphic, free, coincides with the action of a subgroup of $\widetilde{GL^+(2,\mathbb{R})}$ and contains the shifts. The quotient* $\mathrm{Stab}(\mathscr{T})/\mathbb{C}$ *is a complex manifold.*

Proof. Bridgeland's main theorem says that for each connected component $\Sigma \subset \mathrm{Stab}(\mathscr{D})$, there are a linear subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}, \mathbb{C})$, with a well-defined linear topology, and a local homeomorphism $\mathscr{Z} : \Sigma \to V(\Sigma)$ which maps a stability condition $(Z, \mathscr{P})$ to its central charge Z. Holomorphicity thus follows from the holomorphicity of the $\mathbb{C}$-action on a vector space via multiplication by e^z. The action is free since $z * Z = Z$ implies that $e^x = 1$, that is, $x = 0$, and $z * \mathscr{P} = \mathscr{P}$ implies that $y = 0$. Note that the $\mathbb{C}$-action by z has the same effect as the action of $(e^{-x}A, y) \in \widetilde{GL^+(2,\mathbb{R})}$, where A is rotation by the angle $-\pi y$. The shift $[1]$ can be realized at the action of $i\pi \in \mathbb{C}$. We omit the proof of the second sentence. $\square$

Below, we state some facts without proof. (Proofs can be found in [Oka04, Section 4].) Denote by $(\mathbb{Z})$ the copy of $\mathbb{Z}$ that acts on $\mathrm{D^b\,Coh}(\mathbb{P}^1)$ by tensoring with line bundles. Let X be the subset of $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))$ consisting of all stability conditions $(Z, \mathscr{P})$ satisfying the following properties:

(a) $\mathscr{O}$, $\mathscr{O}(-1)[1]$ are semistable,
(b) $\phi(z_0) > 0$,
(c) $\phi(z_{-1}) = 1$ and $m(z_{-1}) = 1$ (m is defined as on page 1).

Then $(\mathbb{Z})\mathbb{C} \cdot X = \mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))$ (so X contains a fundamental domain) and X is isomorphic to the upper half-plane $\mathbb{H}$ by the map sending $(Z, \mathscr{P})$ to $\log(m(z_0)) + i\pi(z_0)$.

A fundamental domain of $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))/(\mathbb{Z})\mathbb{C}$ is isomorphic to $K := \{x + iy \in \mathbb{C} \mid y > 0, \cos y \geq e^{-|x|}\}$, as shaded in Figure 6 below. When passing to $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))/(\mathbb{Z})\mathbb{C}$, one identifies points on the boundary with the same imaginary part.

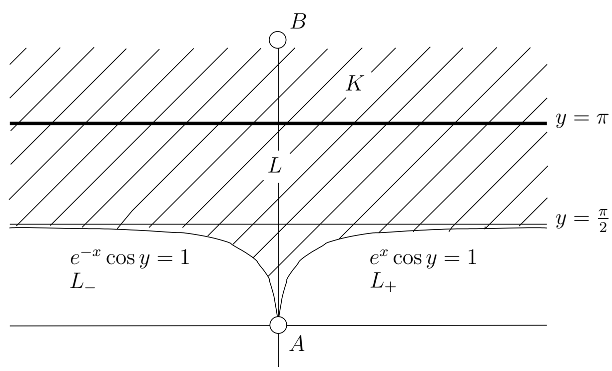


FIGURE 6. A fundamental domain of $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))/(\mathbb{Z})\mathbb{C}$

By the Riemann mapping theorem and the reflection principle, this fundamental domain is conformally equivalent to $\mathbb{C}^*$, as in Figure 7 on the following page. (K_u is the left half of the shaded region and K_l is the right half of the shaded region. We shall abuse notation and label the image under each map by the same letters.)

We can now prove the main result of this talk.

Proof. The action of $(\mathbb{Z})$ on $\mathfrak{X} = \mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))/\mathbb{C}$ gives an exact sequence

$$0 \mapsto \pi_1(\mathfrak{X}) \to \pi_1(\mathfrak{X}/\mathbb{Z}) \xrightarrow{\alpha} \pi_0(\mathbb{Z}) \to \pi_0(\mathfrak{X}).$$

We shall show that $\mathfrak{X}$ is connected. Since $X \cong \mathbb{H}$ is connected, so is $\mathbb{C} \cdot X$. Now, $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1)) = (\mathbb{Z})\mathbb{C} \cdot X$, so it suffices to check that $(\mathbb{Z})$ fixes some connected component of $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))$. This is true because $(\mathbb{Z})$ fixes $\{(Z, \mathscr{P}) \in \mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1)) \mid \mathscr{P}((0,1]) = \mathrm{Coh}(\mathbb{P}^1)\}$, which lies in X. Hence $\mathfrak{X}$ is connected and the map α is a surjective map $\mathbb{Z} \to \mathbb{Z}$, hence an isomorphism. It follows that $\pi_1(\mathfrak{X})$ is 0, so $\mathfrak{X}$ is the universal covering of $\mathbb{C}^*$, that is, $\mathfrak{X} = \mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1))/\mathbb{C} = \mathbb{C}$. Since $H^1(\mathbb{C}, \mathscr{O}) = 0$, there is a unique extension $\mathrm{Stab}(\mathrm{D^b\,Coh}(\mathbb{P}^1)) \cong \mathbb{C}^2$. This completes the proof. $\square$

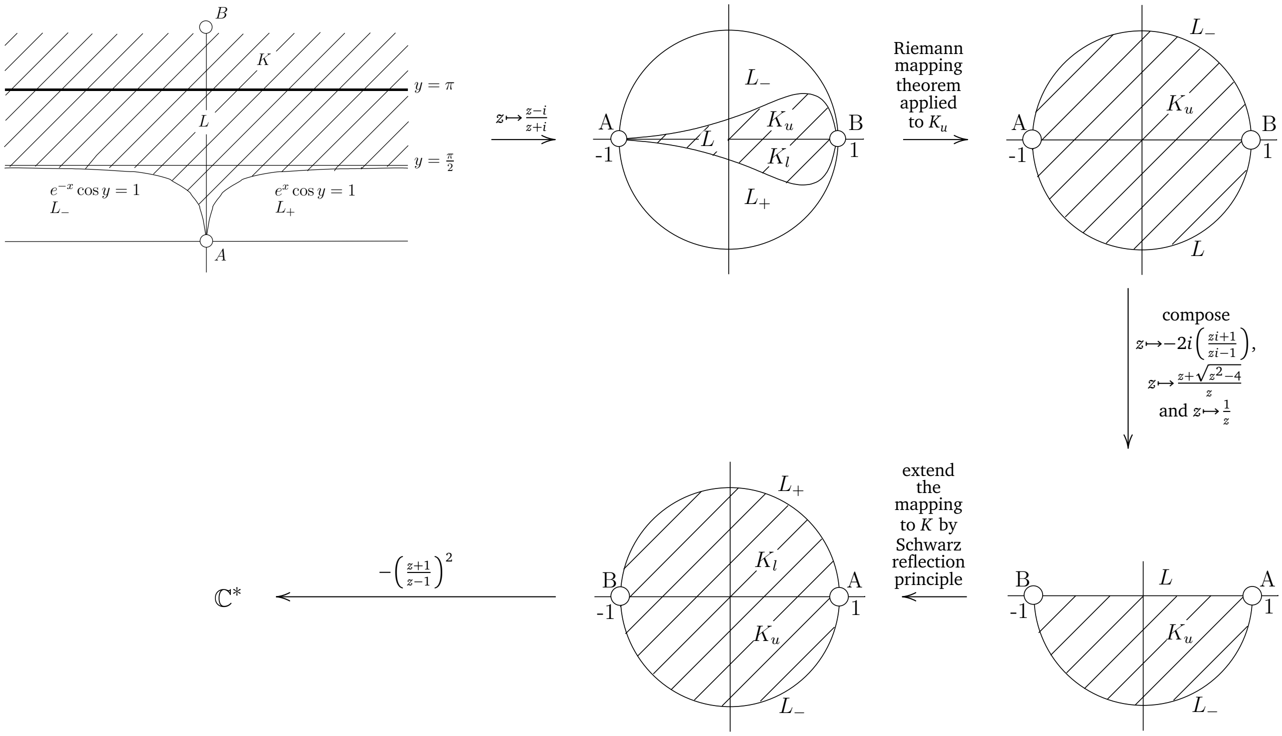


FIGURE 7. Conformal equivalence between $K := \{x + iy \in \mathbb{C} \mid y > 0, \cos y \geq e^{-|x|}\}$ with points on the boundary with the same imaginary part identified and $\mathbb{C}^*$. The diagrams in the right-hand column show how the conformal equivalence acts on K_u.

REFERENCES

[Bri07] T. Bridgeland, Stability conditions on triangulated categories, *Ann. of Math. (2)* **166** (2007), 317–345.

[Mac04] E. Macrì, Some examples of spaces of stability conditions on derived categories, preprint, arXiv:math/0411613v3 [math.AG].

[Oka04] S. Okada, Stability manifold of $\mathbb{P}^1$, preprint, arXiv:math/0411220v3 [math.AG].