Charmaine Stab($\mathbb{P}^1$)

Omar showed that $\forall$ X sm, projective, $g \geq 1$,

$$\mathrm{Stab}(D^b \mathrm{Coh}(X)) \cong \widetilde{GL^+}(2,\mathbb{R}).$$

I'll handle $X \cong \mathbb{P}^1$.

Main Theorem

$$\mathrm{Stab}(D^b \mathrm{Coh}(\mathbb{P}^1)) \cong \mathbb{C}^2$$

as complex manifolds

Strategy: We'll show that the quotient

$$\mathrm{Stab}(\mathbb{P}^1) / \mathbb{C} \times \mathbb{Z}$$

a certain $\mathbb{C} \times \mathbb{Z}$ action

is $\mathbb{C}^\times$.

What's the action? First, $\mathbb{C}$ action

Def (Z,P) stability condition.

Fix $z = x + iy \in \mathbb{C}$. $x \curvearrowright Z$, $y \curvearrowright P$ as follows:

$$z * (Z,P) := (e^{z} Z, P(\phi - y/\pi)).$$

rescaling rotation

This comes from a map

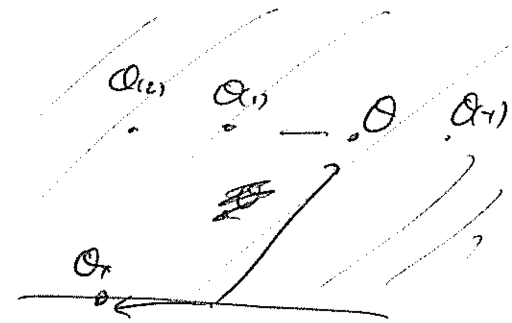
$$\mathbb{C} \longrightarrow \widetilde{GL}^+(2,\mathbb{R}) \curvearrowright \mathrm{Stab}(\mathbb{P}^1).$$

<u>Ex</u> $\sigma \in \mathrm{Stab}(\mathbb{P}^1)$

Is given by choosing

$\cdot\ Z(\mathcal{O}_x) \in \mathbb{R}_{<0} \qquad \mathcal{O}_x = k_x$

and $\cdot\ Z(\mathcal{O}) \in \mathbb{H} \setminus \mathbb{R}$ . .

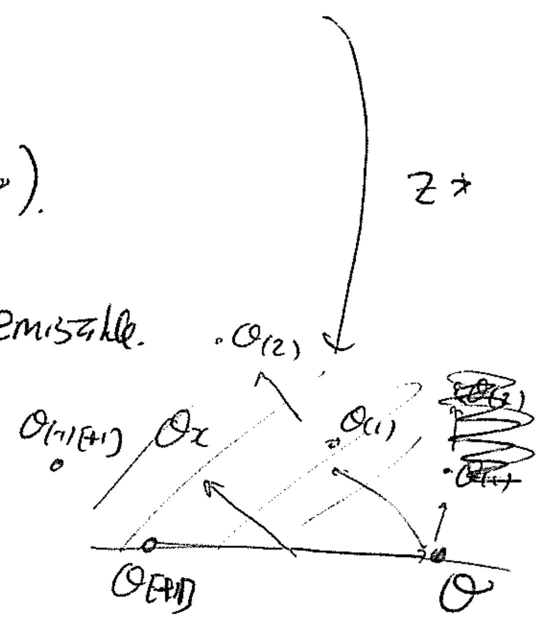


Rotation by $z = -iH(\mathcal{O})$ gives

$$z * (Z, P) = (\overline{Z}, \overline{P})$$

where $\overline{P}((0,2])$ equiv to $\mathrm{Rep}(\circ \rightrightarrows \circ)$.

and all line bundles and torsion sheaves are semistable.



Note $\overline{P}([0,1)) = \langle \mathcal{O}, \mathcal{O}(-1)[1] \rangle$.

Now if we fix the heart and vary Z, what happens to semistable objects?

We find chambers in ~~Stab(P1)~~ $\mathrm{Stab}_{\mathrm{Rep}(\circ \rightrightarrows \circ)}(D\mathrm{Coh}\,\mathbb{P}^1)$

- $\phi(\mathcal{O}) < \phi(\mathcal{O}(-1)[1])$ → Get Z up to reparam of $\mathbb{C}$ by $GL^+(2,\mathbb{R})$, and all shifts of $\mathcal{O}$, $\mathcal{O}(-1)[1]$ are stable.
- $\phi(\mathcal{O}) = \phi(\mathcal{O}(-1)[1])$ → any non-zero object in heart is semistable.
- $\phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[1])$ → only multiples of $\mathcal{O}$, $\mathcal{O}(-1)[1]$ are semistable.

example of a wall.

<u>Def</u> A <u>wall</u> is a codim 1 submfld of Stab s.t. semistable objects change when you pass through it.

Rotation changes $\heartsuit$, but fixes semistables.

Wall-crossing changes semistables, fixes heart.

If we start with

$$\phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[2])$$

$$Z(\mathcal{O}) = z_0, \qquad z_{-1} = Z(\mathcal{O}(-1)[2])$$

if we vary Z s.t. making z_{-1} go below real axis, semistables won't change, but we'll move to different heart:

$$\langle \mathcal{O}, \mathcal{O}(-1)[2] \rangle.$$

$\updownarrow$ corresponds to

$$\mathrm{Rep}(\circ \quad \circ).$$

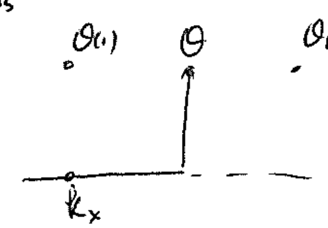
This is an example of $D^b(\text{heart}) \neq D^b \mathrm{Coh}\,\mathbb{P}^1$!

$\boxed{\text{Charmaine}}$ CONT'D.

Example of wall-crossing

Recall: Stability condition on $Coh(\mathbb{P}^1)$, rotated, determined by where we send $\mathcal{O}$ and $\mathcal{O}(-1)[1]$

Standard one is



Rotate:

Get a heart equivalent to $Rep(\bullet \rightrightarrows \bullet)$.

How to see this?

By Beilinson's theorem, we have a functor

$$D^b Coh(\mathbb{P}^1) \longrightarrow D^b Rep(\bullet \rightrightarrows \bullet)$$

$$E \longmapsto Hom^\bullet(\mathcal{O} \oplus \mathcal{O}(1), E)$$

cochain complex

It turns out

$$\mathcal{O} \longmapsto k\langle id_{\mathcal{O}} \rangle$$

$$\mathcal{O}(-1)[1] \longmapsto \cong k\langle id_{\mathcal{O}(-1)} \rangle$$

and let me describe module structure.

$Rep(\bullet \rightrightarrows \bullet) \cong$ left kQ modules

The algebra kQ is the quiver

$$ke_1 \bullet \rightrightarrows \bullet\, ke_2 \oplus kf \oplus kg.$$

under this correspondence

There's an action of $A = End(\mathcal{O} \oplus \mathcal{O}(1))$ on $Hom^\bullet(\mathcal{O} \oplus \mathcal{O}(1), E)$

What is the action?

kQ \ $M = kQ$	e_1	e_2	f	g
e_1	e_1	0	0	0
e_2	0	e_2	f	g
f	f	0	0	0
g	g	0	0	0

The $A \otimes M \to M$ is given by "do m first."

Likewise, let $M = End(\mathcal{O} \oplus \mathcal{O}(1)) \cong k\, id_{\mathcal{O}} \oplus k\, id_{\mathcal{O}(1)} \oplus k x_0 \oplus k x_1$.

$$\mathcal{O} \underset{x_1}{\overset{x_0}{\rightrightarrows}} \mathcal{O}(1)$$

A \ M	$id_{\mathcal{O}(1)}$	$id_{\mathcal{O}}$	x_0	x_1
$id_{\mathcal{O}(1)}$	$id_{\mathcal{O}(1)}$	0	0	0
$id_{\mathcal{O}}$	0	$id_{\mathcal{O}}$	x_0	x_1
x_0	x_0	0	0	0
x_1	x_1	0	0	0

$A \otimes M \to M$ by precomposition! $a \otimes m = m \circ a$.

So quiver is

$$k\, id_{\mathcal{O}(1)} \rightrightarrows k\, id_{\mathcal{O}} \oplus k x_0 \oplus k x_1.$$

So under Beilinson's functor,

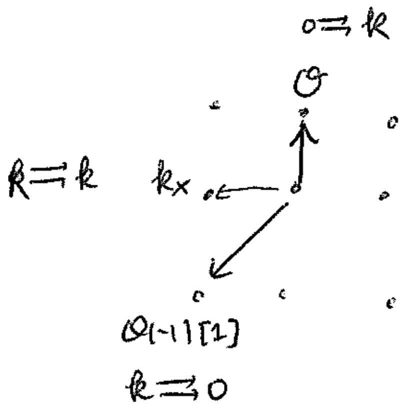
$$\mathcal{O} \longmapsto 0 \rightrightarrows k,$$

And using Serre duality, you can see that

$$\mathcal{O}(-1)[1] \longmapsto k \rightrightarrows 0.$$

(Just compute all the homs from $\mathcal{O} \oplus \mathcal{O}(1)$ to $\mathcal{O}(-1)[1]$.)

So usual stability condition looks like



And we can determine an $\cong$

$$K_0 D^b \mathrm{Coh}\, \mathbb{P}^1 \longrightarrow K_0 D^b \mathrm{Rep}\, Q$$

$$(\deg, \mathrm{rk}) \longmapsto (\dim V_1, \dim V_2) \qquad V_1 \rightrightarrows V_2.$$
$$\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad (d_1, d_2)$$

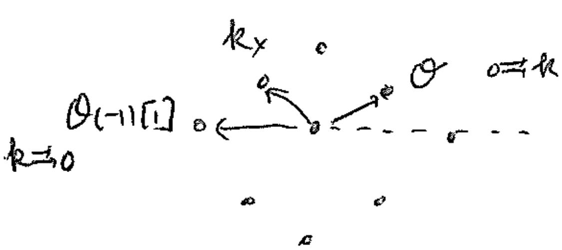
behaves as

$$(0,1) \longmapsto (0,1)$$
$$(1,0) \longmapsto (1,1)$$

So the matrix is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Lets rotate $\mathbb{C}$ by 45 degrees:



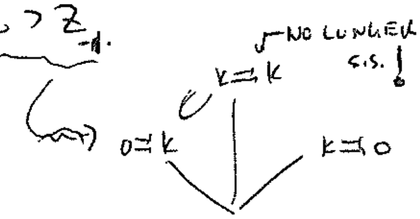
So after this rotation, the heart is generated by $\mathcal{O}$ and $\mathcal{O}(-1)[1]$.

Now let's change the relation btwn $Z(\mathcal{O})$ and $Z(\mathcal{O}(-1)[1])$.

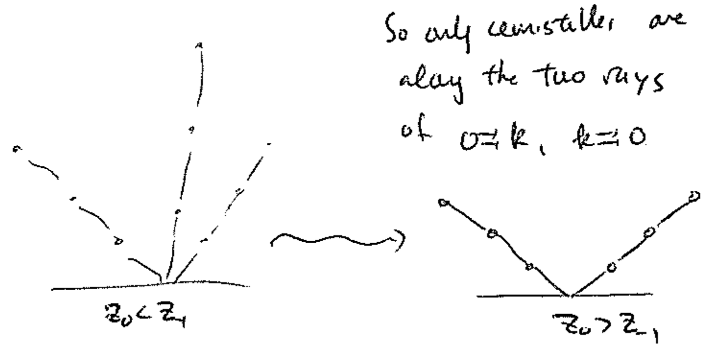
Let $Z_0 = \phi(Z(\mathcal{O}))$
$\quad\; Z_{-1} = \phi(Z(\mathcal{O}(-1)[1]))$

There are three cases:

- $Z_0 < Z_{-1}$
- $Z_0 = Z_{-1}$ ⇝ every object semistable
- $Z_0 > Z_{-1}$.

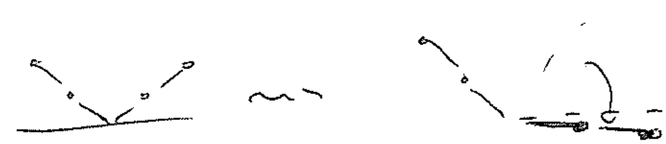


So only semistables are along the two rays of $0 \rightrightarrows k$, $k \rightrightarrows 0$

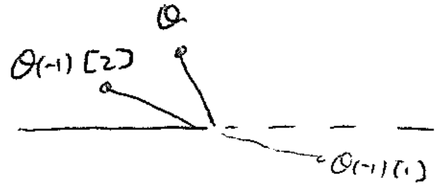


So $Z_0 = Z_{-1}$ is a wall.

Now: If you have $Z_0 > Z_{-1}$, shift Z_{-1} to fall below positive $\mathbb{R}$ axis



Our heart will change.



But in $D^b Coh(\mathbb{P}^1)$,

$$\mathrm{Ext}^\bullet(\mathcal{O}, \mathcal{O}(-1)[2]) \cong 0$$

$$\mathrm{Ext}^\bullet(\mathcal{O}(-1)[2], \mathcal{O}) \cong 0 .$$

So this new heart is equivalent to

$$\mathrm{Rep}(\bullet \quad \bullet) \cong \mathrm{Vect} \oplus \mathrm{Vect}.$$

Note

$$D^b(\mathrm{Vect} \oplus \mathrm{Vect}) \neq D^b Coh(\mathbb{P}^1).$$

Chenglong, intro to ∞-categories

Reference: Jacob Lurie, Higher Topos Theory.

__Defn__ Let $[n]$ be finite, linearly ordered set of $n+1$ elements,

Δ is category of objects $[n]$, morphisms $[n] \to [m]$ order-preserving.

__Defn__ A simplicial set is a functor

$$X : \Delta^{op} \longrightarrow \mathrm{Sets}$$

The category of simplicial sets,

$$\mathrm{Set}_\Delta$$

has morphisms natural transformations.

Lets understand Δ.

There are coface maps

$$d^i : [n-1] \longrightarrow [n]$$

sending

$$\begin{array}{c} 0 \to 1 \to 2 \to \cdots \to n-1 \\ \downarrow \\ 0 \to 1 \to 2 \to \cdots \to i \to \cdots \to n. \end{array}$$

skip i

We also have codegeneracy maps,

$$s^j : [n+1] \longrightarrow [n]$$

$$\begin{array}{c} 0 \to 1 \to 2 \to \cdots \to j \to j+1 \to \cdots \to n+1 \\ \downarrow \\ 0 \to 1 \to 2 \to \cdots \to j \to j+1 \to \cdots \to n \end{array}$$

double up on j.

These satisfy cosimplicial relations

If X is a simp set w/

$$X_n := X([n]),$$

the maps

$$d_i : X_n \to X_{n-1}$$

$$s_j : X_n \to X_{n+1}$$

Satisfy the simplicial relations

$$d_i d_j = d_{j-1} d_i \qquad i<j$$

$$d_i s_j = s_{j-1} d_i \qquad i<j$$

$$d_j s_j = 1 = d_{j+1} s_j$$

$$d_i s_j = s_j d_{i-1} \qquad i > j+1$$

$$s_i s_j = s_{j+1} s_i \qquad i \leq j .$$