$\mathfrak{a}\not{0}x$ (24/10)

Recall: Let X be a smooth projective curve with positive genus $\mathrm{Stab}(X) \cong \widetilde{GL}^+(2,\mathbb{R})$

Lemma. For any numerical stability condition, all line bundles & all sky scraper sheaves are stable.

Sublemma. triangle $A \to E \to B$ in $D^b\mathrm{Coh}X$ and $E \in \mathrm{Coh}X$, $\mathrm{Ext}^{\le 0}(A,B)$ then $A \cong A_0 \oplus A_1[-1]$ $B \cong B_0 \oplus B_{-1}[1]$.

From this triangle we write

$$0 \to B_{-1} \to A_0 \to E \to B_0 \to A_1 \to 0$$

Now we need to prove another sublemma

sublemma. $g(X) > 0$, $\mathrm{Ext}^{\le 0}(A,B) = 0$ and the previous assumptions hold $\Rightarrow A, B \in \mathrm{Coh}X$.

Proof. Have $B_{-1} \to A_0$ since $g(X)$, ω_X has sections and we can make $B_{-1} \to A_0 \otimes \omega_X$

cohomology sheaf

by Serre Duality $Ext^1(A_0, B_{-1}) \neq 0 \Rightarrow Ext^1(A, B[-1]) \neq 0$

$\Rightarrow$ $Ext^0(A, B) \neq 0$. Dual argument show $B_0 \to A_1$

also non zero ---

Recall, Mukai pairing was defined

$$\langle \mathcal{E}, \mathcal{F} \rangle = \sum (-1)^i \dim Ext^i(\mathcal{E}, \mathcal{F})$$

true for elliptic curve,

__Claim.__ $\langle \mathcal{E}, \mathcal{F} \rangle = -\deg \mathcal{E})(\mathrm{rk}\, \mathcal{F}) + (\deg \mathcal{F})(\mathrm{rk}\, \mathcal{E})$

$= -\chi(\mathcal{E})\, \mathrm{rk}(\mathcal{F}) + \chi(\mathcal{F})\, (\mathrm{rk}\, \mathcal{E})$

To prove, note ~~[illegible]~~ this holds for line bundles. Next note that if

$$Ext^i(\mathcal{E}, \mathcal{F}) = Ext^i(\mathcal{O}_X, \mathcal{E}^\vee \otimes \mathcal{F})$$

locally free

Then $\langle \mathcal{E}, \mathcal{F} \rangle = \chi(\mathcal{E}^\vee \otimes \mathcal{F}) = \deg(\mathcal{E}^\vee \otimes \mathcal{F})$

$+ \mathrm{rk}(\mathcal{E})\, \mathrm{rk}(\mathcal{F}) + \chi(\mathcal{O}_X)$

by our definition of degree.

__Comment.__ Looks like the general correct formula

$\langle \mathcal{E}, \mathcal{F} \rangle = \deg \mathcal{F}\, \mathrm{rk}(\mathcal{E}) - \deg(\mathcal{E})\, \mathrm{rk}(\mathcal{F}) + \mathrm{rk}\, \mathcal{E}$

$\mathrm{rk}\, \mathcal{F} (1-g)$

The result follows because it holds for line bundles and ~~[illegible]~~ line bundles generate the numerical Grothendieck group. $\boxtimes$

<u>Proof of lemma.</u> Let $E = \alpha$ line bundle or $E = k_x$ skyscraper sheaf. Look at Harder–Narasimhan filtration, take first object (semi-stable) and form a triangle $A \to E \to B$. Then $\mathrm{Ext}^{\leq 0}(A, B) = 0$ and A, B coherent sheaves, so this is actually a SES in the t-structure

$$0 \to A \to E \to B \to 0$$

If $E = k_x$ done it has no subobjects.

If $E = \alpha$ then A, B torsion this contradicts the assumption.

Look at Jordan-Hölder filtration. A single slice $\mathcal{P}(\phi(E))$ is abelian (~~a~~ interval is quasi-abelian, length 1 interval - abelian again, see Bridgeland paper).

$$E_0 \to E_1 \to \cdots \to E_n = E$$

$S_i = E_i / E_{i-1}$ stable of phase $\phi(E)$

If $\mathrm{Hom}(S_i, E) \neq 0$ then

$$\{ E' \subset E \mid JH \text{ factors of } E' \text{ are } \cong S_i \}$$

Let E' be maximal

$$0 \to E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

That implies $\mathrm{Hom}(E', E'') = 0$ (it has to be zero on all S_i, but E is built from them by extension). This shows you can "~~[illegible]~~ partly rearrange ~~[illegible]~~" Jordan-Holder filtration by putting all factors isomorphic to a fixed S_i in the beginning.

בס״

If E is skyscraper - no stable factors.

$\Rightarrow$ E already stable.

If E line bundle similar to before get

E'' torsion, E' line bundle and no hom's

between them. So it is also stable.

Confusion We are actually talking about

two different t-structure. $P(\phi(E))$ need

not have

make mistake & ignored different t-structure

We get a triangle $E' \to E \to E''$ and

we don't know they are coherent sheaves.

But $\mathrm{Hom}(E', E'') = 0$, $\mathrm{Ext}^{<0}(E', E'') = 0$

by lemma E', E'' are coherent sheaves

and then we can apply the argument

to show they are sub sheaves.

In line bundle potential problem: $E'' = 0$?

But then, by rank argument get

$$[E] = \ell [S_i]$$

But

	degree	rank
k_x	1	0
$\mathcal{L}$	$*$	1

and now is divisible by $\ell \neq 1$! So $\ell = 1 \Rightarrow$ E stable.

Now that we know they are stable, can finish argument for Stab(X). Now we know if we pick $\sigma' \in \mathrm{Stab}(X)$ and look at

$$\mathrm{Hom}(\mathcal{L}, k_x) \neq 0, \ \underset{\mathrm{Hom}(\mathcal{L} \otimes \omega_X^{-1}, k_x)}{\overset{\shortparallel}{\mathrm{Ext}^1(k_x, \mathcal{L})}} \neq 0$$

So $\phi(\mathcal{L}) < \phi(k_x)$ and $\phi(k_x) - 1 < \phi(\mathcal{L})$

That shows that $\sigma' = (Z', P')$

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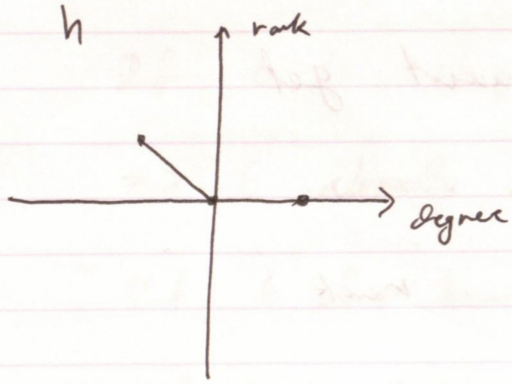
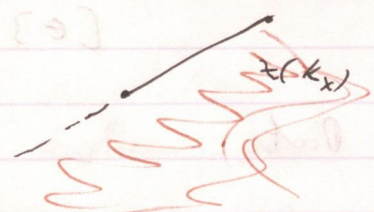


$\xrightarrow{Z'} \mathbb{C}$



Note $Z' : \eta \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R}^2$, so the action has to be free.

Also shows we can find $\sigma'' \in \mathrm{Orb}.\mathbb{Z}(\sigma')$ with

$\sigma'' = (Z, P'')$ (P'' "slightly shifted" slicing)

So the only thing we need to worry is maybe we have a 2 shift in the phase, but the inequality

$$\phi(k_x) - 1 < \phi(d) < \phi(k_x)$$

forces the phase not to have such a structure.

So the heart has all skyscraper and line bundles.

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Now we pause and pass to Charmaine talk about $\mathbb{P}^1$... Ref. Okada "stability mfld on $\mathbb{P}^1$", Arend Bayer "A tour to stability conditions on derived categories" (mini course notes).

Omar has shown for X smooth projective of $g \geq 1$, $\mathrm{Stab}\, D^b\mathrm{Coh}(X) \cong \widetilde{GL}^+(2,\mathbb{R})$. Now:

Main Thm. $\mathrm{Stab}\, D^b\mathrm{Coh}(\mathbb{P}^1) \cong \mathbb{C}^2$ as cplx mfld.

Strategy. Quotient of stability mfld of $\mathrm{Stab}\, D^b\mathrm{coh}(\mathbb{P}^1)$ by a $\mathbb{C}\times\mathbb{Z}$ action is $\mathbb{C}^\times$.

Defn. $\mathbb{C}$-action defined as follows: let (Z, P) be a stability condition and $z = x+iy \in \mathbb{C}$. Then:

$z * (Z, P)$ is defined by $z * Z = e^{z} Z$

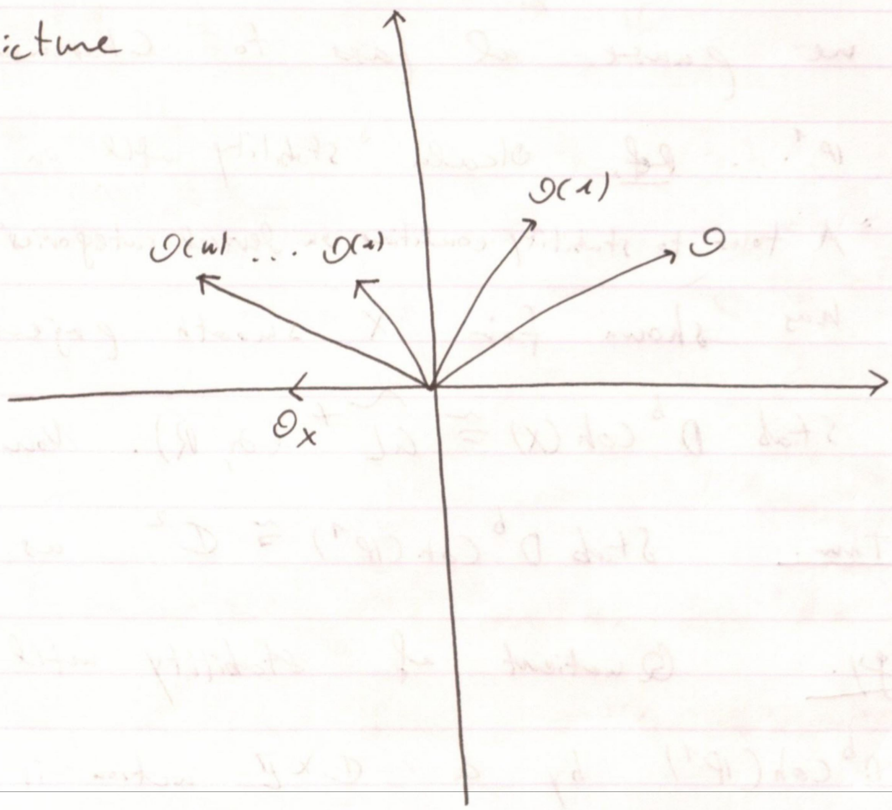
and on the slicing $(z*P)(\phi) = P(\phi - \frac{y}{\pi})$

"rotation".

Ex. stability condition on $D^b(\mathrm{Coh}(\mathbb{P}^1))$ is given by choosing $Z(\mathcal{O}_x) \in \mathbb{R}_{<0}$ and $Z(\mathcal{O}) \in \mathbb{H} \setminus \mathbb{R}$

(want it to have different phase from sky scraper and).

a picture



small amount to rotate a little less...

Rotation by $z = -i\phi(\mathcal{O}) - \varepsilon$ gives

$$z * (Z, P) = (\bar{Z}, \bar{P})$$

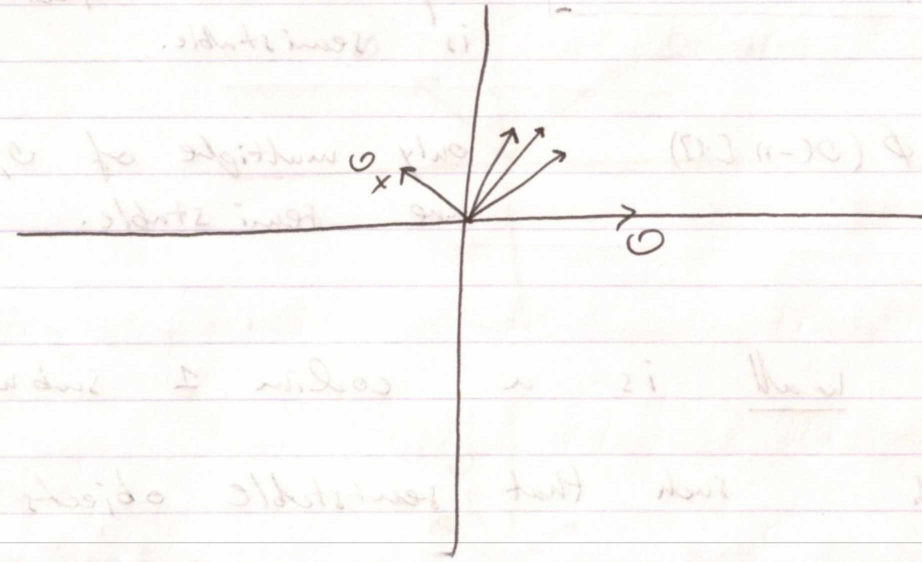
(0,1)?

s.t. $\bar{P}(0,1]$ equiv to $\mathrm{Rep}(\bullet \rightrightarrows \bullet)$ and

all line bundles and torsion sheaves are semistable.

3:02

picture of the rotated stability condition



(Hiro) <u>Comment.</u> negative degree line bundle do not show up on upper half plane only their shift, so need to be careful about what you mean by "line bundle".

maybe $[0,1)$? → $\mathcal{P}(0,1] = \langle \mathcal{O}, \mathcal{O}(-1)[1] \rangle$

If we fix this heart and vary Z what happens to semistable objects? going to get chambers in

$\mathrm{Stab}_{\mathrm{Rep}(\bullet \rightrightarrows \bullet)}(D\,\mathrm{Coh}(\mathbb{P}^1))$. 3 situations:

- $\phi(\mathcal{O}) < \phi(\mathcal{O}(-1)[1])$. Get Z up to reparametrization of $\mathbb{C}$, by an element of $GL^+(2,\mathbb{R})$. All shifts of $\mathcal{O}, \mathcal{O}(-1)[1]$ are stable.

~~$\phi(\mathcal{O}) = \phi(\mathcal{O}(-1)[1])$~~

- $\boxed{\phi(\mathcal{O}) = \phi(\mathcal{O}(-1)[1])}$ any non zero object in heart is semistable.

a wall

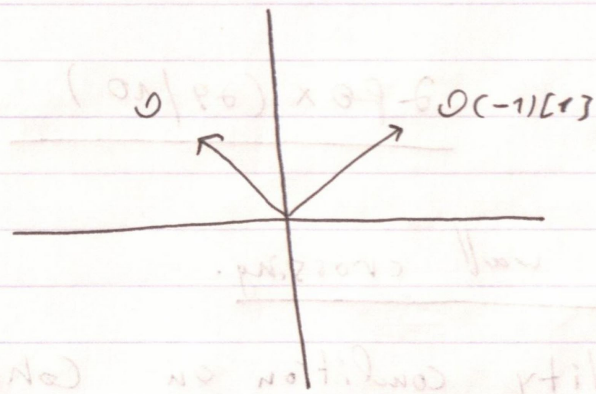
- $\phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[1])$ only multiple of $\mathcal{O}, \mathcal{O}(-1)[1]$ are semi stable.

<u>Defn.</u> A <u>wall</u> is a codim 1 submanifold of Stab such that semistable objects change when you pass through it.

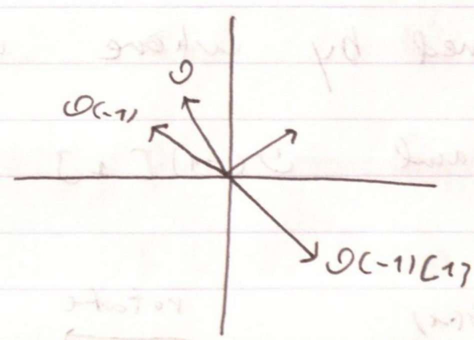
Contrast with "rotation" which changes the heart but keep semistable objects the same

So if we start with $\overset{z_0}{\overset{\shortparallel}{\phi}}(\mathcal{O}) > \overset{z_{-1}}{\overset{\shortparallel}{\phi}}(\mathcal{O}(-1)[1])$ and vary Z by varying z_0 and z_{-1} i.e. let z_{-1} go below $\mathbb{R}$ then semistable objects don't change, but move to another heart, generated by $\langle \mathcal{O}, \mathcal{O}(-1)[2] \rangle \xleftrightarrow[\text{correspond to}]{}$ Rep$(\bullet \quad \bullet)$

Before



After



This is an example of $D^b(\mathrm{Rep}(\bullet \quad \bullet)) \cong D^b(\mathrm{Coh}\,\mathbb{P}^1)$