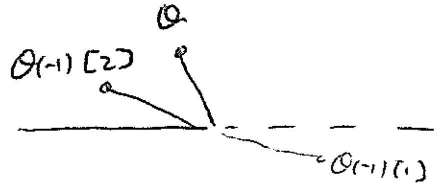


But in $D^b Coh(\mathbb{P}^1)$,

$$Ext^\bullet(\mathcal{O}, \mathcal{O}(-1)[2]) \cong 0$$

$$Ext^\bullet(\mathcal{O}(-1)[2], \mathcal{O}) \cong 0 .$$

So this new heart is equivalent to

$$Rep(\bullet \quad \bullet) \cong Vect \oplus Vect.$$

Note

$$D^b(Vect \oplus Vect) \neq D^b Coh(\mathbb{P}^1).$$

Chenglong, intro to ∞-categories

Reference: Jacob Lurie, Higher Topos Theory.

Defn Let $[n]$ be finite, linearly ordered set of $n+1$ elements,

Δ is category w/ objects $[n]$,

morphisms $[n] \to [m]$ order-preserving.

Defn A simplicial set is a functor

$$X : \Delta^{op} \longrightarrow Sets$$

The category of simplicial sets,

Set_Δ

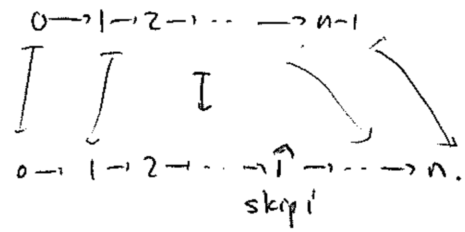
has morphisms natural transformations.

Let's understand Δ.

There are coface maps

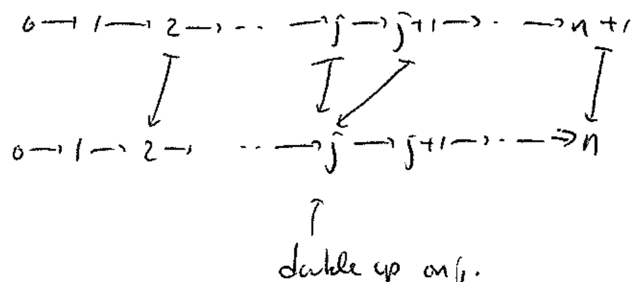
$$d^i : [n-1] \longrightarrow [n]$$

sending



We also have codegeneracy maps,

$$s^j : [n+1] \longrightarrow [n]$$



These satisfy cosimplicial relations

If X is a simp set w/

$$X_n := X([n]),$$

the maps

$$d_i : X_n \to X_{n-1}$$

$$s_j : X_n \to X_{n+1}$$

Satisfy the simplicial relations

$$d_i d_j = d_{j-1} d_i \qquad i<j$$

$$d_i s_j = s_{j-1} d_i \qquad i<j$$

$$d_j s_j = 1 = d_{j+1} s_j$$

$$d_i s_j = s_j d_{i-1} \qquad i>j+1$$

$$s_i s_j = s_{j+1} s_i \qquad i \leq j .$$

Ex. $\Delta^n = \hom(-,[n])$

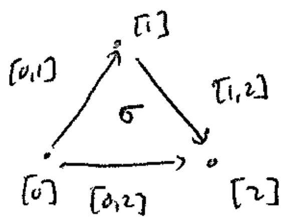
For $n=2$,

$$\Delta^2[0] = [0],\ [1],\ [2]$$

$$\Delta^2[1] = [0,1],\ [1,2],\ [0,2]$$
$$[0,0],\ [1,1],\ [2,2]$$

$$\Delta^2[2] = \underset{\sigma}{\underset{\parallel}{[0,1,2]}},\ \dots\dots$$

These fit together as



and σ in fact generates all elems via d, s.

By Yoneda Lemma,

$$\hom(\Delta^n, X) \cong X_n.$$

Ex Singular chains functor.
Given X space,

$$(\mathrm{Sing}\,X)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X).$$

where $|\Delta^n|$ is the geometric n-simplex,

$$|\Delta^n| = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \text{ s.t. } \textstyle\sum t_i = 1,\ t_i \geq 0\}.$$

A map $[n] \to [m]$ induces a map $|\Delta^n| \to |\Delta^m|$.

$\mathrm{Sing}(X)$ turns out to be a <u>Kan complex</u>.

<u>Defn</u> A simp set X is a <u>Kan complex</u> if $\forall\ 0 \leq i \leq n$, $\forall$ maps

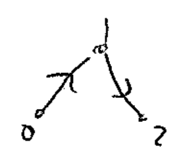
$$\Lambda_i^n \longrightarrow X$$

$\exists$ lift

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Here, $\Lambda_i^n \subset \Delta^n$ is i^{th} horn; it's subsimplicial set of Δ^n generated by $d_k\sigma$ for $k \neq i$.

Ex



$= \Lambda_1^2 \subset \Delta^2$.

Or, Λ_i^n is a colimit over diagrams

$$\begin{tikzcd} \Delta^{n-2}_{k,l} \arrow[r, "d^{l-1}"] \arrow[d, "d^k"'] & \Delta^{n-1}_k \\ \Delta^{n-1}_l & \end{tikzcd} \qquad \forall\ k < l \neq i.$$

Hence $\hom(\Lambda^n_i, X)$ is limit of diagrams

$$\begin{tikzcd} \hom(\Delta^{n-2}_{k,\ell}, X) & \hom(\Delta^{n-1}_k, X) \arrow[l, "d_{\ell-1}"'] \\ \hom(\Delta^{n-1}_\ell, X) \arrow[u, "d_k"] & \end{tikzcd}$$

running over all $k<\ell\neq i$.

So $f: \Lambda^n_i \longrightarrow X$

$$\Updownarrow$$

$(x_0, \dots, \hat{x_i}, \dots, x_n)$, $x_k \in X_{n+1}$

s.t. $d_{\ell-1} x_k = d_k x_\ell \;\; \forall\, k<\ell$

To be Kan means $\exists\, \sigma \in X_n$ s.t. $x_k = d_k \sigma \;\forall k$, given above data.

<u>Prop'n</u> $\mathrm{Sing}(X)$ is a Kan complex.

This gives adjunction

$$\begin{tikzcd} \mathrm{Top}_{cw} \arrow[r, shift left, "\mathrm{Sing}"] & \mathrm{Kan} \arrow[l, shift left, "|\cdot|"] \end{tikzcd}$$

where $|\cdot|$ is geometric realization.

$$|X| = \coprod X_n \times |\Delta^n| \Big/ (x, \theta^* y) \sim (\theta_* x, y).$$

Where $\theta = s_i$ or d_i.

<u>Prop'n</u>

$$\mathrm{Hom}_{\mathrm{Top}_{cw}}(|X|, Y) \cong \mathrm{Hom}(X, \mathrm{Sing}\, Y)$$

Further,

$$|\mathrm{Sing}\, X| \longrightarrow X$$

is a weak homotopy equivalence, as is

$$Y \longrightarrow \mathrm{Sing}\, |Y|.$$

So the point is spaces are Kan complexes, and vice versa.

<u>Nerves</u>. Note $[n]$ can be considered a category since it's a poset.

<u>Defn</u> Given $\mathcal{C}$ category,

$$N(\mathcal{C})_n := \mathrm{Func}([n], \mathcal{C}).$$

So

$$N(\mathcal{C})_0 = \mathrm{obj}\, \mathcal{C}$$
$$N(\mathcal{C})_1 = \{\text{morphism } x \to y\}$$
$$N(\mathcal{C})_n = \{ C_0 \xrightarrow{f_0} C_1 \to \cdots \xrightarrow{f_{n-1}} C_n \}$$

The face maps are given by composing morphisms. Degeneracy maps by inserting identity morphisms.

d_i: $C_0 \to C_1 \to \cdots \to C_n$

$$\downarrow$$

$$C_0 \to \cdots \to C_{i-1} \xrightarrow{f_i \circ f_{i-1}} C_{i+1} \to \cdots \to C_n$$

s_i: $C_0 \to \cdots \to C_i \xrightarrow{id} C_i \to \cdots \to C_n$.

Prop K simp set. Then (1) $\Leftrightarrow$ (2).

(1) $K \cong N(\mathcal{C})$

(2) For $0<i<n$,

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Ex.

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

$\Lambda^2_1 \to N(\mathcal{C})$ is a diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd}$$

Extension to Δ^2 is just composing:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & z \end{tikzcd}$$

This inspires defn of ∞-category.

Defn (quasi-category). S a simp set is called a weak Kan complex if $\forall\ 0<i<n$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & S \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Rmk $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

"Pf":

$$\begin{tikzcd} & y & \\ x \arrow[ur, "f"] \arrow[rr, "id"'] & & x \end{tikzcd}$$

is $\Lambda^2_0 \to N(\mathcal{C})$.

To fill this to Δ^2, we must have an inverse

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "id"'] & & x \end{tikzcd}$$

Motivation: Given space X, define

$\Pi_{\leq\infty}X$ to be "category" where

obj pts of X

1-morphisms paths in X

n-morphisms n-simplices in X

Every hom has an inverse up to homotopy.

We want a notion of $(\infty,1)$ category including both $(\infty,0)$ categories (spaces) and usual categories.

weak Kan condition accomodates both.

$\boxed{\text{Danny}}$ Chenglong told you a ∞-category is a simplicial set S s.t.

$\forall\ 0<i<n$,

$\forall\ f: \Lambda^n_i \longrightarrow X$,

$\exists$ lift

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

But there's another angle to approach homotopical categories.

<u>Outline</u>:

1. Topological Categories
2. Topological Categories $=$ ∞-categories

<u>Defn</u> (Topological Category) is a category enriched over compactly generated, weakly Hausdorff spaces.

So if X, Y are two objects,

$\mathrm{Map}_{\mathcal{C}}(X,Y)$ is a topological space.

We also have an associative composition law: A continuous map

$$\mathrm{Map}_{\mathcal{C}}(X_0,X_1) \times \mathrm{Map}_{\mathcal{C}}(X_1,X_2)$$
$$\downarrow$$
$$\mathrm{Map}_{\mathcal{C}}(X_0,X_2).$$

This strictness actually makes top cats hard to work w/ for some purposes — we'll want to replace ~~homotopy~~ commutative diagrams w/ homotopy coherent diagrams.