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2 Fox (29/10)

Now we pass to ∞-categories. The reference:

Higher topos theory, Chapter 1.

First we'll define a simplicial set.

<u>Defn.</u> Assume $[n]$ is a finite linear ordered set

$$[0\to 1\to 2\to \dots \to n]$$

The category Δ is defined so that

Obj (Δ): $[n]$

Mor (Δ): maps $[n]\to[m]$ which are order preserving

A simplicial set X is a functor $\Delta^{op}\to$ Sets.

We can form a category of simplicial sets

More explicitly, we have a set and

$d^i : [n-1] \to [n]$ coface

$$\left([0\to 1\to \dots \to n-1] \mapsto [0\to 1\to \dots \to \hat{i} \to \dots \to n] \right)$$

drop i

$s^j : [n+1] \to [n]$ codegeneracy

$\xi_0 =$

$$\left(\begin{array}{c} [0 \to \dots \to j \to j+1 \to j+2 \to \dots \to n+1] \\ \downarrow \quad \downarrow \quad \downarrow \\ \mapsto \; [0 \to \dots \to j \to j \to j+1 \to \dots \to n] \end{array} \right)$$

So denote $X_n = X([n])$. We have

face maps $\;\; d_i : X_n \to X_{n-1}, \qquad s_j : X_n \to X_{n+1} \;\;$ degeneracy maps

simplicial identities

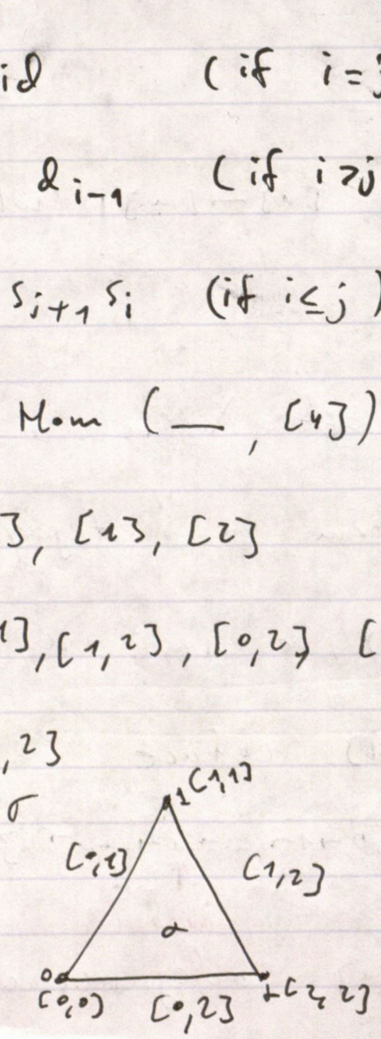
$$\begin{cases} d_i d_j = d_{j-1} d_i & (\text{if } i<j) \\ d_i s_j = s_{j-1} d_i & (\text{if } i<j) \\ d_j s_j = id & (\text{if } i=j \text{ or } i=j+1) \\ d_i s_j = s_j d_{i-1} & (\text{if } i>j+1) \\ s_i s_j = s_{j+1} s_i & (\text{if } i\le j) \end{cases}$$

<u>Example.</u> $\Delta^n = \mathrm{Hom}(\_\_, [n])$. Δ^2:

$\Delta^2[0]$: $[0], [1], [2]$

$\Delta^2[1]$: $[0,1], [1,2], [0,2]$ $[0,0], [1,1], [2,2]$

$\Delta^2[2]$: $[0,1,2]$



Yoneda lemma. for X, $\mathrm{Hom}(\Delta^n, X) \cong X_n$

Example. Simplicial functor, X topological space

$(\mathrm{Sing}\, X)_n = \mathrm{Hom}_{Top}(|\Delta^n|, X)$ where we define

$|\Delta^n| = \{(t_0, \ldots t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\ t_i \geq 0\}$

A map $[n] \to [m]$ induces a map on the geometric simplices $|\Delta^n| \to |\Delta^m|$. $\mathrm{Sing}\, X$ is also special in some sense.

Defn. A Kan complex is a simplicial set s.t.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

existence, not uniqueness

i th horn is a sub simplicial set in Δ^n, generated by $d_k \sigma$ with $k \neq i$.

We can form for $k < l \neq i$

$$\begin{array}{ccc} \mathrm{Hom}(\Delta^{n-2}_{k,l}, X) & \longleftrightarrow & \mathrm{Hom}(\Delta^{n-1}_{k}, X) \\ d^k \updownarrow & & \updownarrow \\ \mathrm{Hom}(\Delta^{n-1}_{l}, X) & \longleftrightarrow & \mathrm{Hom}(\Lambda^n_i, X) \end{array}$$

a morphism $\Lambda^n_i \xrightarrow{f} X \iff$ n-tuple $(x_0, \dots, \hat{x}_i, \dots, x_n)$ with $x_k \in X_{n-1}$. They satisfy $d_{l-1} x_k = d_k x_l$ when $k < l$.

The extension property is equal to $\exists \sigma \in X_n$ s.t. $x_k = d_k \sigma$.

<u>Proposition.</u> $\mathrm{Sing}(X)$ is a Kan complex.

$$\mathrm{Top}_{cw} \underset{\text{geometric realization}}{\overset{\mathrm{Sing}}{\rightleftarrows}} \mathrm{Kan}$$

$\overset{=}{302}$

$\theta_i^* = d_j$ or $s_k \cdots$

Defn. Geometric realization functor is defined as

$$X \longrightarrow |X| = {}^{X_n \times |\Delta^n|} \Big/ \begin{array}{l}(x, \theta^* y) \sim \\ (\theta_* x, y)\end{array}$$

Prop. $\mathrm{Hom}_{\mathrm{Top}_{CW}}(|X|, Y) \cong \mathrm{Hom}(X, \mathrm{Sing}\, Y)$

and $|\mathrm{Sing}\, X| \to X$ is a w.e. and

for Kan complex we get $Y \to |\mathrm{Sing}\, Y|$ a homotopy

equivalence.

So studying space is equivalent to studying

Kan complexes

Defn. Given a category $\mathcal{C}$ we define the

nerve to be

$$N(\mathcal{C})_n := \mathrm{Func}([n], \mathcal{C}).$$

i.e. $N(\mathcal{C})_0 = \mathrm{obj}(\mathcal{C})$

$N(\mathcal{C})_1 = \mathrm{Mor}(\mathcal{C}) = \{x \to y\}$

$N(\mathcal{C})_n = c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \to \cdots \xrightarrow{f_n} c_n$

degeneracy and face maps:

$$c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n$$

$$d_i \quad c_0 \to c_1 \to \cdots \to c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \to \cdots \to c_n$$

$$s_i \quad c_0 \to c_1 \to \cdots \to c_i \xrightarrow{id_{c_i}} c_i \to \cdots \to c_n$$

<u>Prop.</u> $K \in sSet$. Then the following are equivalent

(1) $K \cong N(\mathcal{C})$ for a category $\mathcal{C}$

(2) For $0 < i < n$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

So we can identify nerves of categories.

<u>Illustration.</u> Consider

$$\begin{tikzcd} & 1_y \arrow[dr, "g"] & \\ 0_x \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & 2_z \end{tikzcd}$$

can compose uniquely

$\Longleftrightarrow$

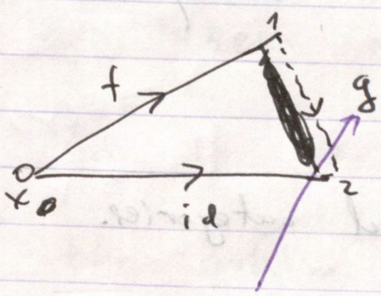
$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Defn. A quasi-category or weak Kan complex means a simplicial set S with lifting

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

for $0<i<n$ (in Kan complex, it holds for $0 \le i \le n$)

Remark. $N(\mathcal{C})$ is a Kan complex $\Leftrightarrow$ $\mathcal{C}$ is groupoid because we consider the identity map



extension exist because

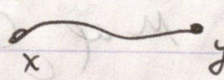
$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$f \circ g = id$. So we can invert any morphism.

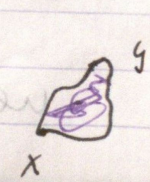
Motivation. Top space X

$\pi_{\le\infty} X$ has objects points of X

1-morphism



2-morphism homotopy



This is called ∞-groupoid. Its an $(\infty, 0)$ category.

Summary. We've just learned about 3 things.

① Kan complex - think of as spaces.

② Simplicial set can encode all categories

③ Weak Kan complex - want to encompass all this things.

Outline of next week. ① Topological categories.

② Topological categories are the same as quasicategories.

where Defn. A topological category is a category enriched over (compactly generated weak Hausdorff) spaces. So if $X, Y \in obj(\mathcal{C})$ then $Map_{\mathcal{C}}(X, Y)$ is a topological space, we have an associative composition law i.e.

there is a continuous map

$$Map_{\mathcal{C}}(x_0, x_1) \times Map_{\mathcal{C}}(x_1, x_2) \times \cdots \times Map_{\mathcal{C}}(x_{n-1}, x_n) \longrightarrow Map_{\mathcal{C}}(x_0, x_n)$$

$\partial f \partial x$ (31/10)

This is the 2^{nd} in the series. Last time we explained what a top. category is.

Now we need to answer

Q when are two topological categories equivalent?

So if we have $\mathcal{C}, \mathcal{D}$ we can't set the condition of iso between mapping spaces. This turns out to be two strong. So we want to define a "homotopy" cat of top categories.

historically, the name is reserved to another thing

Defn. For a top. cat $\mathcal{C}$, we define $h\mathcal{C}$

$ob(h\mathcal{C}) = ob(\mathcal{C})$, $x, y \in ob(\mathcal{C})$

$Map_{h\mathcal{C}}(x, y) = [Map_{\mathcal{C}}(x, y)]$ cw complex