Dold-Kan correspondence

face operator ∂_i

degeneracy σ_i

Homotopy group ~~of~~ of Kan complex

base point $* \in X_0$ $\quad * = \sigma_0^{n}(*) \in X_n$

$Z_n = \{x \in X_n \mid \partial_i x = * \quad \forall i\}$

$y, y' \in Z_n$

$y \sim y'$ if $\exists x \in X_{n+1}$ s.t. $\partial_i x = \begin{cases} * & 0 \le i \le n-1 \\ y & i = n \\ y' & i = n+1 \end{cases}$

Prop $\quad X$ Kan complex, then $\sim$ equivalence relation.

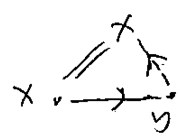
$\pi_n(X) = Z_n/\sim$ homotopy group.

self-reflexive $\quad \partial_n \sigma_n(x) = \partial_{n+1} \sigma_n(x) = x$

transitive :

$\in X_0$ $\quad x \sim y \quad y \sim z \Rightarrow x \sim z$

symmetric



$x \sim y \Rightarrow y \sim x$.

X top space $\quad \pi_n(\mathrm{Sing}\, X) = \pi_n(X)$

X Kan complex $\quad \pi_n(X) = \pi_n(|X|)$

X simplicial group (always a Kan complex in this case)

$N_n(X) := \{x \in X_n \mid \partial_i x = * \quad \forall i \quad 0 \le i \le n-1\}$

$0 \xleftarrow{\partial_0} N_0(X) \xleftarrow{\partial_1} N_1(X) \xleftarrow{\partial_2} N_2(X) \leftarrow \cdots$

$H_n(\;\cdot\;) = \pi_n(X)$

X simplical object in an abelian cat A

$N_n(X) := \bigcap_{i=0}^{n-1} \ker d_i \subseteq X_n$ normalized chain complex of A

$$\leftarrow N_{n-1}(X) \xleftarrow{\partial_n} N_n(X) \xleftarrow{\partial_{n+1}} N_{n+1}(X) \leftarrow$$

$\pi_n(X) := H_n(N_*(X))$

If A = Cat of abelian groups or R-mod

two defs coincide

$C(X)$ $\quad X_{n-1} \xleftarrow{\sum_i (-1)^i d_i} X_n \xleftarrow{\sum_i (-1)^i d_i} X_{n+1}$

unnormalized chain complex

$D(X)_n := \sum_i \sigma_i(X_{n-1}) \subseteq X_n$ degenerate chain complex

Prop $X_n = N_n(X) \oplus D_n(X)$ and $D(X)$ acyclic hence

$\pi_n(X) = H_n(N(X)) = H_n(C(X))$

A abelian cat

Dold-Kan Thm. $N: \mathrm{Fun}(\Delta^{op}, A) \to Ch_{\geq 0}(A)$ is a equivalence of cat. $\pi_n(X) = H_n(N(X))$

homotopic maps between simplicial obj correspond to chain hamotopic map between chain complexes.

construct inverse functor $DK: Ch_{\geq 0}(A) \to \mathrm{Fun}(\Delta^{op}, A)$

C chain complex

$$X_n = \bigoplus_{p \leq n,\ [n] \overset{\eta}{\twoheadrightarrow} [p]} C_p[\eta]$$

$C_p[\eta]$ is a copy of C_p

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \lambda \downarrow & & \downarrow \eta \\ [q] & \xrightarrow{\beta} & [p] \end{array}$$

want to define map $X_n \to X_m$ $\quad C_p[\eta] \to X_m$

If $q = p$ (i.e. $\eta \alpha$ surjective) $\quad C_p[\eta] \xrightarrow{=} C_p[\lambda]$

If $q+1 = p$ $\quad \begin{smallmatrix} 0 & 1 & \cdots & q \\ \downarrow & \downarrow & \beta & \downarrow \\ 0 & 1 & \cdots & q & q+1=p \end{smallmatrix}$ $\quad C_p[\eta] \to C_q[\lambda]$

$$C_p \xrightarrow{d} C_{p-1}$$

Simple example

Eilenberg-MacLane space

G abelian group

$0 \leftarrow G \leftarrow 0 \leftarrow 0 \leftarrow$

$0 \quad 1$

$X_0 = 0$

$X_1 = G$

$X_2 = \bigoplus_{[2] \overset{\eta}{\twoheadrightarrow} [1]} G[\eta] = G_1 \oplus G_2$

η_0: $0, 1 \mapsto 0$, $2 \mapsto 1$ $\qquad$ η_1: $0 \mapsto 0$, $1, 2 \mapsto 1$

$[1] \xrightarrow{\varepsilon_0, \varepsilon_1, \varepsilon_2} [2] \xrightarrow{\eta} [1]$

$\partial_i : G_1 \oplus G_2 \to G$

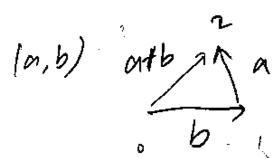
$\partial_0 \quad (a, b) \to a$

$\partial_1 \quad (a, b) \to a+b$

$\partial_2 \quad (a, b) \to b$

geometric realization

$\bigvee_G S^1$



after glue (a,b) 2-cell $\qquad S^1_a + S^1_b \sim S^1_{a+b}$

Proof of Dold-Kan

$N DK(C_\cdot) \cong C_\cdot \qquad \Rightarrow \qquad N$ fully and essentially surjective

$X_n = \bigoplus_{p \le n} \bigoplus_{[n] \overset{\eta}{\twoheadrightarrow} [p]} C_p[\eta] \qquad C_p[\eta] \cong C_p$

If $p \neq n \qquad [n] \twoheadrightarrow [p] \qquad$ $[n] \to [n-1]$ $\qquad$ $[n] \to [n-1]$, $[p] \xrightarrow{\cong} [p]$ $\qquad C_p[\eta] \subseteq O_i(X_{n-1})$

$X_n = N_n(X) \oplus D_n(X)$

$N_n(X) = \quad C_n[id] = C_n$

Need to show N faithful

$f. \; X \to Y \qquad N(f) = 0 \quad \Rightarrow \quad f = 0$

$$\begin{tikzcd}
 & X_n = N(X)_n \oplus D_n(X) \arrow[r, "f"] & Y_n = N(Y)_n \oplus D_n(Y) \\
X_{n-1} \arrow[ur, "\sigma_i"] \arrow[rr, "f=0"] & & Y_{n-1} \arrow[u, "\sigma_i"]
\end{tikzcd}$$