there is a continuous map

$$\mathrm{Map}_{\mathcal{C}}(x_0, x_1) \times \mathrm{Map}_{\mathcal{C}}(x_1, x_2) \times \text{---}$$

$$\text{---} \times \mathrm{Map}_{\mathcal{C}}(x_{n-1}, x_n) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x_0, x_n)$$

<u>∂ß∂x (31/10)</u>

This is the 2^{nd} in the series. Last time we explained what a top. category is.

Now we need to answer

<u>Q</u> when are two topological categories equivalent?

So if we have $\mathcal{C}, \mathcal{D}$ we ~~[illegible]~~ can't set the condition of iso between mapping spaces. This turns out to be two strong. So we want to define a "homotopy" ~~[illegible]~~ cat of top categories.

historically, the name is reserved to another thing

Defn. For a top. cat $\mathcal{C}$, we define ~~[illegible]~~ $h\mathcal{C}$

$ob(h\mathcal{C}) = ob(\mathcal{C})$, $x, y \in ob(\mathcal{C})$

cw complex

$\mathrm{Map}_{h\mathcal{C}}(x, y) = [\mathrm{Map}_{\mathcal{C}}(x, y)]$

Defn. An equivalence between $\mathcal{C}$ and [redacted] $\mathcal{D}$ such that $h\mathcal{C}$ and $h\mathcal{D}$. More concretely:

~~Defn. An equivalence between $\mathcal{C}$ and $\mathcal{D}$~~

Fully faithful. $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\text{w.e.}} \mathrm{Hom}_{\mathcal{D}}(F(X, F(Y))$

Essentially surjective. For any object $X \in \mathcal{D}$, X equivalent to $F(Y)$ for some $Y \in \mathrm{ob}\,\mathcal{C}$.

there exists a morphism $f: X \to F(Y)$ $g: F(Y) \to X$ such that $f \circ g \simeq \mathrm{id}$, $g \circ f \simeq \mathrm{id}$ (i.e. paths in mapping space)

Q Can you always find an inverse equivalence?

A In ∞-cat should be yes.

In this model no! We'll later pass to a model [redacted] where it's true.

To show $\mathcal{C}at_{top}$ are equivalent to ∞-categories we introduce $\mathcal{C}at_{\Delta}$

quasi-categories

simplicial categories

There ~~...~~ is an adjoint pair

$$sSet \underset{Sing}{\overset{|\cdot|}{\rightleftarrows}} Top$$

Can apply

$$\text{(2)}\ Sing(X) \longleftarrow\!\shortmid\ X\ \text{(1)}$$

$$\longmapsto |Sing X| \overset{w.e.}{\simeq}\ \text{(3)}$$

So to turn $\mathcal{C} \in Cat_{\Delta}$ to top. cat. just apply geometric realization to each mapping space

$$\mathcal{C} \in Cat_{\Delta} \longmapsto |\mathcal{C}| \in \mathcal{C}at_{Top}$$

Similarly

$$Sing\, D \in Cat_{\Delta} \longleftarrow\!\shortmid\ D \in \mathcal{C}at_{top}$$

Define $h\mathcal{C} := h|\mathcal{C}|$.

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To complete the "proof" we need to show why $\mathcal{C}at_\Delta$ is equivalent to

$\mathcal{C}at_\Delta \xrightarrow{N^{h\mathcal{C}}} sSet$

simplicial nerve i.e. homotopy coherent nerve

If you recall $\mathcal{C}at \xrightarrow{N} sSet$

$Hom_{sSet}(\Delta^n, N\mathcal{C}) = Hom_{\mathcal{C}at}(\Delta^n, \mathcal{C})$

for simplicial nerve this is the same

replace by $N^{h\mathcal{C}}$

$0 \to 1 \to 2 \to \dots \to n$ gets replaced by "thickening" $\mathfrak{C}[\Delta^n]$

Where $\mathfrak{C} : sSet \underset{N^{h\mathcal{C}}}{\overset{\mathfrak{C}}{\rightleftarrows}} \mathcal{C}at_\Delta : N^{h\mathcal{C}}$ ~~adjoint pair~~

Quillen adjunction.

We define the adjoint $\mathfrak{C}$:

$$\Delta^n : \underset{0}{\bullet} \to \underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet} \to \cdots \to \underset{n}{\bullet}$$

$$\mathfrak{C}(\Delta^n) : \underset{0}{\bullet} \quad \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \underset{n}{\bullet}$$

<u>Simplicial category</u> $\quad \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = N(P_{ij})$

where P_{ij} is the poset containing all the elements

$\mathrm{hom}(0,0) = N(P_{00})$

obj $\mathfrak{C}[\Delta^0] = \{0\}$

where P_{00} is poset with elemet $S \subseteq \{0\}$ containing $i=0$, $i=j \Rightarrow$ just 0-simplex

<u>Picture.</u>

$$(N\mathcal{C})_0 = \mathrm{Map}_{sSet}(\Delta^0, N\mathcal{C}) = \mathrm{Fun}_{Cat_\Delta}(\mathfrak{C}[\Delta^0], \mathcal{C})$$

$$= \mathrm{ob}\,\mathcal{C}$$

obj $\mathfrak{C}[\Delta^1] = \{0,1\}$

$$(N\mathcal{C})_1 = \mathrm{Map}_{sSet}(\Delta^1, N\mathcal{C}) = \mathrm{Fun}_{Cat_\Delta}(\mathfrak{C}[\Delta^1], \mathcal{C})$$

$\mathrm{hom}(0,0) = \mathrm{hom}(1,1) = \Delta^0$

$$= \coprod_{\substack{X,Y \in \\ \mathrm{ob}\,\mathcal{C}}} \mathrm{hom}_{\mathcal{C}}(X,Y)_0$$

$\mathrm{hom}(0,1) = N P_{01} = \Delta^0$

because subset $S \subset [0,1]$, contain $0,1$...

$\mathrm{hom}(1,0) = \phi$

$\Rightarrow \underset{0}{\bullet} \!-\!\!-\!\!-\!\!-\! \underset{1}{\bullet}$

$$(N\mathcal{C})_2 =$$

$\vdots$

Now what's $\mathcal{C}[\Delta^2]$?

obj $\mathcal{C}[\Delta^2] = \{0,1,2\}$

repeating the same reasoning as before

$$\left.\begin{array}{l} \hom(0,0)\\ \hom(1,1)\\ \hom(2,2)\end{array}\right\} \cong \Delta^0$$

$$\left.\begin{array}{l} \hom(0,1)\\ \hom\end{array}\right\} \cong \Delta^0$$

__But__ $\hom_{\mathcal{C}[\Delta^2]}(0,2) = NP_{02}$. P_{02} has 2 elements $\{0,2\}$ and $\{0,1,2\}$ and one relation $\{0,2\} \subset \{0,1,2\}$. So
$$\begin{tikzcd} \circ_{h_{\{0,2\}}} \arrow[d] \\ \circ_{h_{\{0,1,2\}}} \end{tikzcd}$$

$\hom(0,2) \cong \Delta^1$

So what does $f \in \mathrm{Fun}_{cat}(\mathcal{C}[\Delta^2], \mathcal{C})$?

It's

$$\begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h_{\{0,2\}}"] & & X_2 \end{tikzcd}$$
$$\underbrace{\hspace{6em}}_{h_{\{0,1,2\}}}$$

what does composition mean

$$\begin{array}{ccccc}
\hom(1,2) & \times & \hom(0,1) & \longrightarrow & \hom(0,2) \\
\parallel & & \parallel & & \\
\Delta^0 & & \Delta^0 & & \\
\{1,2\} & & \{0,1\} & \longmapsto & \{0,1,2\}
\end{array}$$

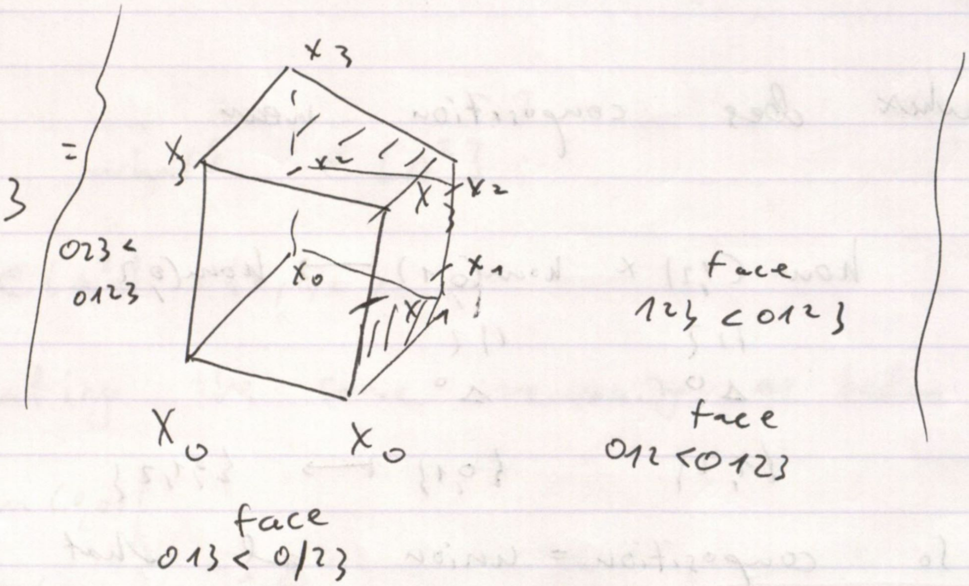
So composition = union and what we get

is

$$(N\mathcal{C})_2 = \left\{ \begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr, "h_{\{0,2\}}"] \arrow[rr, bend right=50, ""{name=B, above}, "g \circ f = h_{\{0,1,2\}}"'] & & x_2 \arrow[from=B, to=1-2, phantom, "\Uparrow"] \end{tikzcd} \right\} =$$

$$\begin{tikzcd} x_2 \arrow[r, dash, "id"] & x_2 \\ & x_1 \arrow[u, dash, "g"'] \\ x_0 \arrow[uu, dash, "h_{\{0,2\}}"] \arrow[r, dash, "id"'] \arrow[ur, phantom, "\Rightarrow"] & x_0 \arrow[u, dash, "f"'] \end{tikzcd}$$

Similarly,

$(N\mathcal{C})_3 =$ 

So $N\mathcal{C}$ basically encodes homotopy coherent diagrams.

So we've just defined $\mathfrak{C} : \underset{\text{cat}}{\infty\text{-}} \longrightarrow \mathrm{Cat}_\Delta$

We can define $hS = h\mathfrak{C}(S)$ for S an ∞-cat.

Now let's talk about mapping spaces in ∞-categories.

$$\mathrm{Map}_{\mathcal{C}}(X,Y) = \mathrm{Map}_{h\mathcal{C}}(X,Y)$$

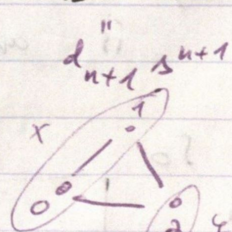
<u>Goal</u>. Find a simplicial set that represents $\mathrm{Map}_{h\mathcal{C}}(X,Y)$.

Obvious first candidate $\mathrm{Map}_{\mathfrak{C}\mathcal{C}}(X,Y)$.
but this is strictly associative and not neccesarily a Kan complex.

<u>Right morphism space</u>. $\mathrm{Hom}^R_{\mathcal{C}}(X,Y)$ where

$$\mathrm{Hom}_{sSet}(\Delta^n, \mathrm{Hom}^R_{\mathcal{C}}(X,Y)) = \left\{ \begin{array}{l} z : \Delta^{n+1} \longrightarrow \mathcal{C} \text{ s.t.} \\ z|_{\Delta^{\{n+1\}}} = y \\ z|_{\Delta^{\{0,1,\dots,n\}}} = x \end{array} \right\}$$

Example.



very very useful to compute $\pi_0, \pi_1 \dots$

<u>Prop</u>. $\mathrm{Hom}^R_{\mathcal{C}}(X,Y)$ is w.e. to $\mathrm{Map}_{h\mathcal{C}}(X,Y)$. This is a Kan complex.

we basically take π_0, this is what we do when we derive a category in homological algebra...

One last thing to say, let's talk about the actual homotopy category (take 3 ☺).

(Denoted $ho\,\mathcal{C}$ or $\pi_0\mathcal{C}$ usually).

When S is an ∞-category

$\boxed{ob(hoS) = ob(S)}$ That's easy. Now we define morphism

<u>Defn.</u> $\Delta^1 \longrightarrow S$, i.e. $\underset{c}{\circ} \xrightarrow{f} \underset{c'}{\circ}$ a morphism.

<u>Defn.</u> Given two morphisms $f,g : \Delta^1 \to S$ we define f homotopic to g if $\exists\, \Delta^2 \to S$

$$\begin{tikzcd} & c' \arrow[dr, "id"] & \\ c \arrow[ur, "f"] \arrow[rr, "g"'] \arrow[rr, phantom, bend left=20, "\Delta^2"] & & c' \end{tikzcd}$$

One annoying thing: distinguished unit in ∞-cat.

in Kan complex notion of htpy is equiv relation

<u>Prop.</u> If S is an ∞-category, then $\sim$ is an equiv relation. <u>Proof.</u> Mapping space is Kan... □

So

$$\boxed{Hom_{hoS}(X,Y) = \text{htpy classes of maps } X \to Y \Big/ \sim}$$

Now will take over.

Equivalence. S ∞-cat. We form hoS:

obj $\mathrm{Hom}(\Delta^0, S)$

mor $\mathrm{Hom}(\Delta^1, S)/\sim$

Def. f is called an equivalence in S if f induces an isomorphism in $ho(S)$.

(Joyal) Prop. Let $\phi : \Delta^1 \to \mathcal{C}$. ϕ is equiv iff for all $f : \Lambda^n_0 \to \mathcal{C}$ s.t. $f|_{\Delta\{0,1\}} = \phi$, f has an extension to Δ^n.

∞-groupoid. $\mathcal{C}$ ∞-cat. If $ho(\mathcal{C})$ is a groupoid, then we'll call $\mathcal{C}$ an ∞-groupoid.

same as Kan complex

Prop (Joyal) TFAE

(1) $h\mathcal{C}$ groupoid

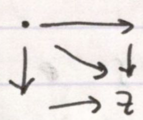
(2) $\Lambda^n_i \subset \Delta^n$, $0 \le i < n$

(3) $\Lambda^n_i \subset \Delta^n$, $0 < i \le n$

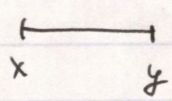
(4) $\Lambda^n_i \subset \Delta^n$, $0 \le i \le n$

think about monoid with only left inverses... this is similar problem and intuition.

<u>Defn.</u> $F : K \to \mathcal{C}$ is called a <u>K-shaped</u> diagram. We define colimit by universal property



We can rephrase by introducing $\mathcal{C}_{F/}$ under category

obj 

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colimit is just initial object in under category.