280x (5/11)

As we discussed the previous time,

we need to define $\star$ — we use Lurie's quasicategories throughout

Defn. Let S, S' be two ∞-categories. Define the *join* as a simplicial set

$$(S \star S') = S_n \cup S'_n \coprod_{i+j=n-1} S_i \times S'_j$$

can be written more compactly if you use the convention negative simplices are point.

Example. $S = \Delta^1$, $S' = \Delta^1$ then $S \star S' = \Delta^3$.

Prop. $S \star S'$ ∞-category.

Proof. $\Delta^i \star \Delta^j = \Delta^{i+j+1}$ so

$$\mathrm{Hom}(\Delta^n, S \star S') = \coprod \mathrm{Hom}(\Delta^i, S) \times \mathrm{Hom}(\Delta^j, S)$$

etc.

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Defn. ① Given $\rho : K \to S$ a map between simplicial sets, we define the under category

$$\mathrm{Hom}_{set_\Delta}(Y, S_{\rho/}) = \mathrm{Hom}_\rho(K \star Y, S)$$

with $f|_K = \rho$ then $\mathrm{Hom}(\Delta^n, S_{\rho/}) = \mathrm{Hom}_{f|_K = \rho}(K \star \Delta^n, S)$.

② Let $\underline{S} = N(\mathcal{C})$ with $\mathcal{C}$ category, and $\rho : K \to \mathcal{C}$. Then ρ induces a map of simplicial sets $\rho : N(K) \to N(\mathcal{C})$. Then:

$$\mathrm{Hom}(\Delta^0, S_{\cdot/\rho}) = \mathrm{Hom}_{f|_{N(K)} = \rho}(N(K) \star \Delta^0, \underline{S})$$

③ The same applies to over category. For maps of simplicial sets $\rho : K \to S$ we define $S_{\cdot/\rho}$ to be defined by the equation

$$\mathrm{Hom}_{set_\Delta}(Y, S_{\cdot/\rho}) = \mathrm{Hom}_{f|_K = \rho}(Y \star K, S_{\cdot/\rho})$$

Prop. If S is an ∞-category then $S_{p/}$, $S_{/p}$ are ∞-categories.

Now we define limits and colimits

Defn. Let S be an ∞-category. We say that X is initial $\Leftrightarrow$ $\mathrm{Hom}^R(X, Y)$ is contractible for $Y \in \mathrm{obj}(S)$. X is terminal $\Leftrightarrow$ $\mathrm{Hom}^R(Y, X)$ is contractible for $Y \in \mathrm{obj}(S)$.

Defn. Let $p : K \to S$ with $K \in \mathrm{Set}_\Delta$, S ∞-category. We can form $S_{p/}$. Then the colimit of p is the initial object (if exists) in $S_{p/}$. Limit of p is the final object in $S_{/p}$.

$\overset{=}{3}\text{o}\,\partial$

Now we'll talk about ~~[illegible]~~

<u>Dold-Kan correspondence</u>

We've defined Kan complexes in the previous week. ~~[illegible]~~ Let X simplicial set, with <u>base</u> point $* \in X_0$.

∂_i face map

σ_j degeneracy map

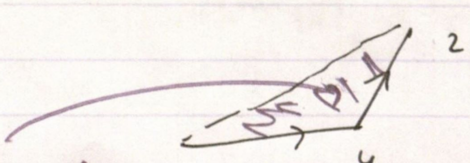
by abuse of notation denote

$* = \sigma_0^i(*) \in X_i$. Given $x, y \in X_n$ we say $x \sim y$ if $\exists\, z \in X_{n+1}$ s.t. $\partial_i z = \begin{cases} * & 0 \le i \le n-1 \\ x & i = n \\ y & i = n+1 \end{cases}$

<u>Prop.</u> If X is Kan then $\sim$ is an equivalence relation.

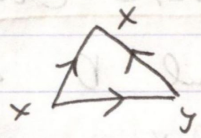
<u>Proof.</u> ① $x \sim x$ $\quad \partial_n \sigma_0(x) = x = \partial_{n+1} \sigma_0(x)$

② If $x \sim y$, $y \sim t$ we have



use Kan condition

③ $x \sim y \Rightarrow y \sim x$. Because



Defn. X Kan, then define $\pi_n(X) =$

$\{x \in X_n,\ \partial_i x = *\}/\sim$.

- Z top. Then $\pi_n(\mathrm{Sing}\, Z) \cong \pi_n(Z)$
- X Kan. Then $\pi_n(X) \cong \pi_n(|X|)$

- If X is a simplicial abelian group. Consider 3 chain complex:

① $N_n(X) = \{x \in X_n \mid \partial_i x = 0 \text{ for } 0 \le i \le n-1\} = \bigcap_{i=0}^{n-1} \ker \partial_i$

$$0 \leftarrow N_0(X) \xleftarrow{\partial_1} N_1(X) \xleftarrow{\partial_2} N_2(X) \leftarrow \cdots$$

Then $H_n(N_\cdot(X)) = \pi_n(X)$. It is called normalized chain complex.

② Can also define complex C

alternating sum of face map

$$C: \cdots \leftarrow X_{n-1} \leftarrow X_n \leftarrow X_{n+1}$$

this is the normal complex of singular chains.

A third complex is D with $D_n = \{ \sigma_i(X_{n-1}) \}$.

It is called the degenerate chain complex.

Prop. $C = D \oplus N$ and D acyclic. So

$$H_n(C) = \pi_n(X) = H_n(N)$$

More generally, in any abelian category

Thm (Dold-Kan) A abelian category

$$\mathrm{Fun}(\Delta^{op}, A) \underset{DK}{\overset{N}{\rightleftarrows}} Ch_{\geq 0}(A)$$

is an equivalence of categories. Moreover

$$\pi_n(X) = H_n(N(X))$$

N preserve homotopies (we didn't really properly define homotopies to discuss that rigorously).

~~Example~~

Defn. The inverse functor

$$DK : ch_{\geq 0}(A) \longrightarrow \mathrm{Fun}(\Delta^{op}, A)$$

is defined in the following way:

let $0 \leftarrow c_0 \leftarrow c_1 \leftarrow c_2 \leftarrow \dots$

$$X_n = \bigoplus_{\substack{p \le n, \\ \eta : [n] \twoheadrightarrow [p]}} C_p[\eta] \quad \text{with } C_p[\overset{n}{\eta}] \text{ a copy of } C_p.$$

Define map α^* for $\alpha : [m] \to [n]$: take

$$\alpha^* : X_n \longrightarrow X_m$$

by defining $\alpha^* : C_p[\overset{n}{\eta}] \to X_m$ as

$$\begin{tikzcd} {[m]} \arrow[r, "\alpha"] \arrow[d, two heads, "\varepsilon"'] & {[n]} \arrow[d, two heads] \\ {[q]} \arrow[r, tail] & {[p]} \end{tikzcd}$$

$$C_p[\eta] \to C_q[\varepsilon] \le X_m$$

(1) If $q = p$ then

$$\begin{tikzcd} \alpha^* : C_p[\eta] \arrow[r] \arrow[d, phantom, "\wr\shortparallel"] & C_q[\varepsilon] \arrow[d, phantom, "\shortparallel"] \\ C_p \arrow[r, "id"] & C_p \end{tikzcd}$$

(2) If $p = q+1$ then $C_p[\eta] = C_p$, $C_q[\varepsilon] = C_q$

$[q]$ $\quad 0, -, q$

$[p]$ $\quad 0, -, q, \underset{p}{\underset{\shortparallel}{q+1}}$

(3) α^* for all other cases.

To get a feel for this construction, we study a simple example.

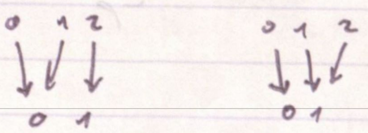
<u>Example.</u> Take $K(G,1)$. Then

$X_0 = \{0\}$

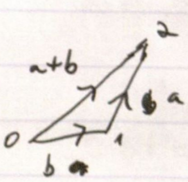
$X_1 = G$

$X_2 = G \oplus G$

study $[2] \twoheadrightarrow [1]$.



The face maps $\underset{G \oplus G}{\overset{\|}{X_2}} \to \underset{G}{\overset{\|}{X_1}}$ by $\begin{cases} \partial_0(a,b) \mapsto a \\ \partial_1(a,b) \mapsto a+b \\ \partial_2(a,b) \mapsto b \end{cases}$



$\overset{=}{3}0$?

have $N(DK(C)) \cong C \Rightarrow N$ fully faithful

and essentially surjective. This is nice.

$$DK(C)_n = \bigoplus_{[n] \twoheadrightarrow [p]}^{\eta} C_p[\eta]$$

If $p < n$ then
$$\begin{tikzcd}
{[n]} \arrow[r, "\alpha"] \arrow[d] & {[n-1]} \arrow[d, "\beta"] \\
{[p]} \arrow[r] & {[p]}
\end{tikzcd}$$
gives

$\alpha^* : C_p[\beta] \xrightarrow{id} C_p[\eta]$. In fact $\alpha^* = \partial_i$.

$C_p[\eta] \leq \sum \sigma_i(X_{n-1})$

$X_n = N(X)_n \oplus D(X)_n$

$N(DK(C)) = C$

also N faitful (i.e. $N(f) = 0 \Rightarrow f = 0$).

$$\begin{tikzcd}
& X_n = N(X)_n \oplus D(X)_n \arrow[rr, "f"] & & Y_n = N(X)_n \oplus D(X)_n \\
X_{n-1} \arrow[ur, "\sigma_i"] \arrow[rr, "f=0"] & & Y_{n-1} \arrow[ur, "\sigma_i"] &
\end{tikzcd}$$

so we get equiv of categories.

Preview for Thursday: we'll talk about DG-cat and nerves. We'll work over fixed commutative ring k. The main theorem of the talk is going to be

DG-nerve

Thm. There is a canonical equiv of ∞-category

$$N_{DG} : \left\{ \begin{array}{c} \text{pretriangulated} \\ \text{DG-cat over } k \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} k\text{-linear stable} \\ \infty\text{-categories} \end{array} \right\}$$

Goal. understand N_{DG} on object level.

(first define the object). Bonus: functorial behavior. We won't prove N_{DG} is an equiv.