∂ fox (7/11)

k = commutative ring (at some point today a field because it is easier).

Thm. There is a canonical equiv of ∞-categories

$$N_{DG} : \left\{ \begin{matrix} \text{pretriangulated DG} \\ \text{categories over } k \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} k\text{-linear stable} \\ \infty\text{-category} \end{matrix} \right\}$$

(Co) Chain complex

Defn. <u>(co)chain complex</u> is a $\mathbb{Z}$-graded k-module

$$C = \bigoplus_{n \in \mathbb{Z}} C^n$$

with a degree 1 endomorphism d such that $d^2 = 0$.

Defn. A <u>morphism</u> of chain complexes: degree 0 graded map which commute with d.

k-com = abelian cat of chain complexes

Defn. $f : C \to D$ morphism. A <u>homotopy</u> $h : f \to g$ is a degree -1 map $h : C \to D$ such that

$hd + dh = g - f.$

Defn If we have a complex $C^\cdot$, define

$$C[m]^n = C^{m+n}$$

with differential $(-1)^m d$. This is the shift.

Defn. for $f : C \to D$ define the

cone $\mathrm{Con}(f) = C[1] \oplus D$ with differential

$$\begin{bmatrix} -d_C & 0 \\ f[1] & d_D \end{bmatrix}$$

That looks unmotivated, so let's talk a little about motivation. They are actually ~~homotopy~~ shifted homotopy kernels i.e.

$$\text{morphism } \mathrm{Con}(f) \longrightarrow E.$$

$$\Updownarrow$$

morphism $D \to E$, with a nullhomotopy of $C \to E$.

Moreover $\mathrm{Con}(f)[-1]$ is a homotopy kernel.

Example. $f: M \to N$ a homomorphism.

M, N = k-modules (complexes in degree 0)

$H^0(\mathrm{con}(f)) = \mathrm{coker} f, \quad H^{-1}(\mathrm{con}(f)) = \ker f.$

We want to define tensor product for two complexes

Def. $$C \otimes D = \bigoplus_{i,j \in \mathbb{Z}} C^i \otimes D^j = \bigoplus_{n \in \mathbb{Z}} \underbrace{\bigoplus_{i+j=n} C^i \otimes D^j}_{n^{\text{th}} \text{ graded piece}}$$

The differential is defined by

$$d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy$$

can see this by sphere smashing...

$\Rightarrow$ monoidal structure on k-com.

Comment (Hiro). All this definitions can be easily motivated by considering CW-complexes and their cellular chain complexes.

There is a braiding as part of the monoidal str

$$C \otimes D \xrightarrow{\sim} D \otimes C$$

with crazy $x \otimes y \longmapsto (-1)^{\deg(x)\deg(y)} y \otimes x$.

$\Rightarrow$ get a symmetric monoidal structure.

Comment (Omar). You can define different braiding by changing the sign of the differential. Probably the easiest example with 2 different braiding

form graded k-module (not complexes) and take ~~[illegible]~~ braiding with Koszul sign rule and without.

$\forall C \in K\text{-com}$ we really want that

$D \mapsto D \otimes C$ has a right adjoint.

Defn. $\mathrm{Hom}^n(C, D) = \{$ degree n graded map $\}$

internal Hom

בס״ד

If $f \in \mathrm{Hom}^n(C,D)$ we define the differential $df = f \circ d_C - (-1)^n d_D \circ f$.

Note that $\mathrm{Hom}(C,D) = Z^0 \mathrm{Hom}^\bullet(C,D)$

zero cocycles

Another thing worthy to point out is that a homotopy $h : f \to g \iff h \in \mathrm{Hom}^{-1}(C,D)$ such that $dh = g - f$

differential in Hom complex

$H^0 \mathrm{Hom}^\bullet(C,D) = \{$ homotopy classes of morphism $C \to D \}$.

That's it. Let's move to the next part.

DG categories

Defn. A DG cat is a category which is enriched over k-com (which makes sense because it is a monoidal category).

stands for **D**ifferential **G**raded

because we want composition in enriched category to extend to tensor products $\mathrm{Hom}^\bullet(Y,Z) \otimes \mathrm{Hom}^\bullet(X,Y) \to \mathrm{Hom}^\bullet(X,Z)$

Defn. A DG-functor is an enriched functor i.e. it is required to induce morphism of $\mathrm{Hom}^\bullet$ complexes.

Defn Say $\mathcal{C}, \mathcal{D}$ are DG-categories and $F, G : \mathcal{C} \to \mathcal{D}$ are DG-functors. A $\psi : F \to G$ is called graded of degree n if for any object $X \in \mathcal{C}$, $\eta_X \in \mathrm{Hom}^n(F(X), G(X))$.

Now $\mathrm{Hom}^\bullet(F,G)$ actually form a complex with differential defined by the same formulas as before.

the category of DG-functors is itself a DG-category

Define. Let $\mathcal{C}$ be a DG-cat. The opposite category $\mathcal{C}^{op}$ has the same objects and function complexes $\mathrm{Hom}^\bullet_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}^\bullet_{\mathcal{C}}(Y,X)$ but composition of $X \leftarrow Y$, $Y \leftarrow Z$

$$\mathrm{Hom}^\bullet(Z,Y) \otimes \mathrm{Hom}^\bullet(Y,X) \xrightarrow{\sim}$$

used braiding crucially

$$\mathrm{Hom}^\bullet(Y,X) \otimes \mathrm{Hom}^\bullet(Z,Y) \longrightarrow \mathrm{Hom}(Z,X)$$

Comment. The braiding on chain complexes is actually symmetric

$$C \otimes D \xrightarrow{\sim} D \otimes C \xrightarrow{\sim} C \otimes D$$

is $\mathrm{id}_{C\otimes D}$. This implies $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

<u>Example.</u> A DG category with one object X is just a DG-algebra $\mathrm{End}^\bullet(X)$. (Can also see directly from algebra).

monoid object in chain complexes.

<u>Defn.</u> $Z^0\mathcal{C}$ = DG-category with same objects, morphisms $X \to Y = Z^0 \mathrm{Hom}(X, Y)$

in this case it is the underlying category

<u>Example.</u> $k\text{-com}_{DG}$ = DG category of chain complexes (morphism are complexes).

$Z^0(k\text{-com}_{DG}) = k\text{-com}$

<u>Defn.</u> $H^0\mathcal{C}$ = homotopy category of DG-category $\mathcal{C}$: a category with same objects, and morphisms $X \to Y = H^0 \mathrm{Hom}^\bullet(X, Y)$.

Defn. A DG-functor $F: \mathcal{C} \to \mathcal{D}$ is called quasi fully faithful provided that F induces quasi-isomorphism on chain complexes. It is quasi-essential surjective if the induced functor on homotopy categories

$$H^0F : H^0\mathcal{C} \to H^0\mathcal{D}$$

It is a quasi equivalence if both hold.

in ∞-cat terminology you just basically drop "quasi"...

Example. Let $\mathcal{A}$ = abelian category. Define $\mathcal{A}\text{-com}_{DG}$ = DG category of complexes in $\mathcal{A}$.
$H^0(\mathcal{A}\text{-com}_{DG}) = \mathcal{H}\mathcal{A}$ = homotopy category of complexes in $\mathcal{A}$.

$\mathcal{A}$-com$_{Dg}$ is an "enhancement" of $\mathcal{H}\mathcal{A}$

Can we come up with Dg-enhancement of the derived category? We have no time for this today. But it is in the notes.

<u>Observation.</u> $\mathcal{H}\mathcal{A} = H^0(\mathcal{A}\text{-com}_{Dg})$ is triangulated

That is not always true. A = nonzero Dg algebra

$H^0 A$ is <u>not</u> triangulated. no zero object, no shift clearly...

We introduce the notion of pretriangulated category.

Defn. Let's say $X \in \mathcal{C}$, then $X[m]$ is the object of $\mathcal{C}$ ~~which~~ that represents the Dg-functor $Y \mapsto \mathrm{Hom}^\bullet(Y, X)[m]$

If it exists, it is called a <u>shift</u>.

<u>Defn.</u> Let $f: X \to Y$ in $\mathcal{C}$. Cone(f) is the object which represents $Z \mapsto \mathrm{Con}\left(\mathrm{Hom}^\bullet(Z,X) \xrightarrow{f\circ -} \mathrm{Hom}^\bullet(Z,Y)\right)$ if it exists. We call it the <u>cone</u>.

<u>Defn.</u> A DG-category $\mathcal{C}$ is called strongly pretriangulated provided that:

- it has a zero object
- all shifts of all objects
- all cones for all morphisms

<u>Claim.</u> For any DG-cat $\mathcal{C}$ there exists a strongly pretriangulated hull $\mathcal{C}^{pt}$ which has the following universal property: for any DG-functor $\mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ strongly pretriangulated, there is an extension $\mathcal{C}^{pt} \to \mathcal{E}$, unique up to unique DG-isomorphism.

strong, kind of a strict construction

Description of construction.

Yoneda $\longrightarrow$ $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$

$\searrow \tilde{\mathcal{C}} \checkmark$

similar to Yoneda in enriched setting $\longrightarrow$ $\underset{\text{dg-cat}}{\mathcal{C}} \to \mathrm{dg\,fun}(\mathcal{C}^{op}, k\text{-comm})$

${}^{c}\mathcal{C}^{et}$

closure under cones + shifts

Defn. $\mathcal{C}$ is called pretriangulated (as opposed to strong pretriangulated) provided that $H^0\mathcal{C} \to H^0\mathcal{C}^{et}$ is an equivalence

Prop. If $\mathcal{C}$ is pretriangulated $\Rightarrow$ $H^0\mathcal{C}$ is triangulated.

∞

Da- nerve

given

$$\underset{\text{Da- cat}}{\mathcal{C}} \xrightarrow[\text{want}]{} \underset{\infty\text{-cat}}{N_{Da}(\mathcal{C})}$$

0- simplices of $N_{Da}(\mathcal{C})$ are objects of $\mathcal{C}$

1- simplices of $N_{Da}(\mathcal{C})$ are morphisms in $\tau^{\circ}\mathcal{C}$

2- simplex in $N_{Da}(\mathcal{C})$ consists of

$$f_1 : X \to Y$$

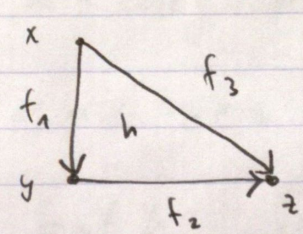
$$f_2 : Y \to Z$$

$$f_3 : X \to Z$$

toghether with a homotopy

$$h : f_2 \circ f_1 \to f_3$$

Illustration.



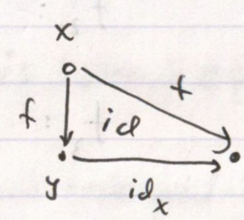
... similarly for n-simplices, $n \geq 2$.

face maps $N_{Du}(\mathcal{C})_1 \longrightarrow N_{Du}(\mathcal{C})_0$ send a morphism to its source & target (there are two face maps). There is a single degeneracy map $N_{Du}(\mathcal{C})_0 \longrightarrow N_{Du}(\mathcal{C})_1$

$$\underset{\text{object}}{x} \longmapsto \underset{\text{identity}}{id_x}$$

3 face maps $N_{Du}(\mathcal{C})_2 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} N_{Du}(\mathcal{C})_1$ send a triangle to its edge.

Two degeneracy maps $N_{Du}(\mathcal{C})_1 \rightrightarrows N_{Du}(\mathcal{C})_2$

$f : X \to Y \qquad \longmapsto$ 

etc.