בס״ד

280x (11/11)

Continue from last time:

Construction (DG-Nerve)

$k\text{-com} \xrightarrow{\tau_{\le 0}} k\text{-com}_{\le 0} \xrightarrow{DK} \mathrm{Fun}(\Delta^{op}, k\text{-mod})$

$\longrightarrow \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$

is lax monoidal.

$\mathcal{C} = DG$ - category $\rightsquigarrow \mathcal{C}_\Delta$ = simplicial category

$N(\mathcal{C}_\Delta)$ ∞ - category.

Prop. There is an equivalence

$$N(\mathcal{C}_\Delta) \xrightarrow{\sim} N_{DG}(\mathcal{C})$$

Fact. There is a canonical triangulated equivalence $H^0\mathcal{C} \xrightarrow{\sim} \mathrm{Ho}\, N_{DG}(\mathcal{C})$.

Now: Quick intro to stable ∞ - category

Defn. An ∞ - category $\mathcal{C}$ is called pointed if it has a zero object (i.e. an object which is both initial and terminal. That means that $\mathrm{Hom}_{\mathcal{C}}(0, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, 0)$ are contractible for all $X \in \mathcal{C}$

Comment. $\mathrm{Fun}(\Delta^1, \mathcal{C})$

$\downarrow$ fiber gives mapping space.

$(x,y) \in \mathcal{C} \times \mathcal{C}$

discussion of left/right mapping spaces

Defn. A triangle is

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

A pullback triangle is a fiber sequence.

A pushout triangle is a cofiber sequence.

Defn A pointed ∞-category $\mathcal{C}$ is called stable provided that:

i) every morphism admits a fiber and cofiber.

ii) a triangle is a fiber sequence

$\iff$ it is a cofiber sequence.

think about SES
in abelian categories.

So think about
distinguished triangle
in a heart. This
is just a SES.

Let's say $\mathcal{C}$ pointed admits cofibers.

Define an endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$

called <u>suspension</u> with $\Sigma X = 0 \amalg_X 0$ i.e.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

Similarly define $\Omega : \mathcal{C} \to \mathcal{C}$ called <u>loop</u>

$\Omega X = 0 \times_X 0$ i.e.

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Σ left adjoint to Ω always.

$\mathcal{C}$ stable $\Rightarrow$ mutually inverse equiv.

<u>Thm.</u> A pointed ∞-category $\mathcal{C}$ is stable

Hard thm.

$\Leftrightarrow$ $\mathcal{C}$ admits finite colimits and

Σ is an equivalence.

<u>Prop.</u> $\mathcal{C}$ = DG-category $\Rightarrow$ $N_{DG}(\mathcal{C})$ stable.

<u>Proof.</u> Assume $\mathcal{C}$ strictly pretriangulated.

So $X \oplus Y = \mathrm{Con}(X[-1] \xrightarrow{0} Y)$.

Cones in $\mathcal{C}$ $\rightsquigarrow$ compute cofiber in

$N_{DG}(\mathcal{C}) \Rightarrow$ $\mathcal{C}$ has finite colimits.

something in HTT

$X \mapsto X[1]$ suspension $\rightsquigarrow$ $\Sigma : N_{DG}(\mathcal{C}) \to N_{DG}(\mathcal{C})$

Thus $X[1] = \mathrm{Con}(X \mapsto 0)$.

invertible!

So by the ~~thm~~ theorem above

we are done. $\blacksquare$

Enrichment

$\exists$ notion of symmetric monoidal ∞-category

It's kind of a pain to write down. There are

$\mathbb{1} \in \mathcal{C}$, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

with lots of homotopy coherence conditions...

Anyway for such a structure $\text{Ho}\,\mathcal{C}$ is symmetric monoidal in the classical sense.

There's also a notion of a monoid object (also called algebra object).

Example. Sp = symmetric monoidal ∞-cat of spectra.

Commutative algebra object in Sp = E_∞-ring.

$A \in \mathcal{C}$ commutative algebra object

under some hypothesis on $\mathcal{C}$ (e.g. $\mathcal{C}$ cocomplete)

essentially need to be able to do Bar-construction

can form symmetric monoidal ∞-category $\text{Mod}_A(\mathcal{C})$ of modules.

Example. Consider k as a discrete E_∞-ring $\rightsquigarrow$ $Sp_k = Mod_k(Sp)$. This is a very abstract defn of module spectra.

Thm. There is a canonical equivalence of symmetric monodial ∞-category.

$$Sp_k \xrightarrow{\sim} N_{DG}(\mathcal{D}(k\text{-mod}))$$

DG-enhancement of the derived category. If k is a field this is just the dg-cat of chain complexes.

Defn. A k-linear stable ∞-category is a stable ∞-category enriched over Sp_k

comment: How does this relate to Lurie's notion of enrichment? UNKNOWN.

precise definition of enrichment is Def. 4.2.1.28 in HA. It is a mess...

Consider first cocomplete stable ∞-categories. Enough to show tensored over Sp_k.

(If your original DG-category sits

$$\mathcal{C} \hookrightarrow Fun(\mathcal{C}^{op}, k\text{-}com_{DG}) = \mathcal{C}^{op}\text{-mod}.$$

practical way to get enrichment

and it inherits the enrichment)

Now Hiro talks about stable ∞-category.

Review.

- ∞-cat $\rightsquigarrow$ homotopy coherent diagram.
- (co) limits = (initial) / final object of (under) over categories.

Assume $\mathcal{C}$ has zero object (i.e.

)

Example. $\mathcal{C} = \text{Chain}_k$. To think of this as ∞-cat either use Dold-Kan etc.

What's zero object? $0 \in \mathcal{C}$. Why?

$$\hom(0, X)_n = \hom_{ch}(0 \otimes C(\Delta^n), X) = *$$

chain maps

chain complex computing homology of Δ^n

pointed spaces i.e. $* \to X$

Example. $\mathcal{C} = \text{spaces}_*$. Here it's easy to say what it means as ∞-cat. There is an object $* \xrightarrow{id} *$.

This is the zero object $(* \xrightarrow{id} *) = 0$ because

$$\hom(0, X) = *$$

$$\hom(X, 0) = *$$

Example. If $\mathcal{C}$ is the nerve of an ordinary category, then zero object for $\mathcal{C}$ is a zero object for the original category.

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Defn. Fix $f: X \to Y$ in $\mathcal{C}$.

A fiber (also known, slightly misleadingly as ~~kernel~~ kernel) for f is a limit to

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

why is called a fiber? because

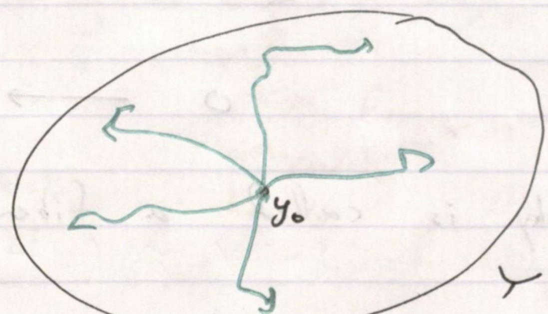
$$\begin{array}{ccc} f^{-1}(*) & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow f \\ * & \longrightarrow & Y \end{array}$$

But always remember our fibers are homotopy fibers.

<u>Example</u> $\mathcal{C} = \text{spaces}_*$. Let $X = 0$.

$$\operatorname{holim}\left(\begin{tikzcd} & 0 = X \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}\right) = \lim\left(\begin{tikzcd} & P_* Y \arrow[d, "ev_1"] \\ 0 \arrow[r] & Y \end{tikzcd}\right)$$

where $P_* Y = \{ \gamma : [0,1] \to Y \mid \gamma(0) = y_0 \}$



a contractible space. Now this is a fibration.

$$= \begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & P_* Y \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

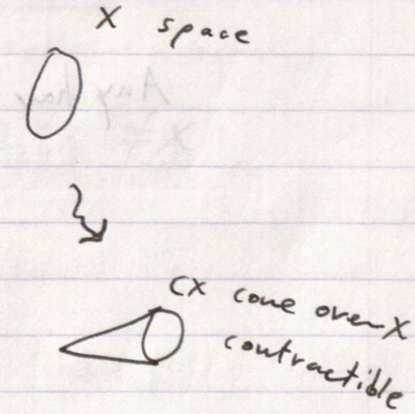
what is that? The loop space

$$\Omega Y = \{ \gamma : [0,1] \to Y \mid \gamma(0) = \gamma(1) = y_0 \}$$

Defn. A cofiber (co limit) of $f: X \to Y$ is a colimit of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

Example. $\mathcal{C}$-Spaces. $Y = *$

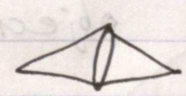


$$\mathrm{hocolim}\left(\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}\right) =$$

cofibration (inclusion)

$$\mathrm{colim}\left(\begin{tikzcd} X \arrow[r] \arrow[d] & CX \\ 0 & \end{tikzcd}\right) = \Sigma X$$

What is ΣX? The suspension is two glued copies of the cone.



ΣX

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<u>Example.</u> $\mathcal{C} = \mathrm{Chain}_k$

cofiber$(X \xrightarrow{f} Y) \simeq$ Mapping cone of f

$\simeq X[1] \oplus Y$ (with differential

$$\begin{pmatrix} -d_C & 0 \\ f[1] & d_D \end{pmatrix}$$)

Anyhow, cofiber$(X \to 0) \simeq X[1]$.

fiber$(X \xrightarrow{f} Y) \simeq$ shift of mapping cone.

fiber$(0 \to X) \simeq X[-1]$.

<u>Defn.</u> $\mathcal{C}$ is stable if

(0) $\mathcal{C}$ has a zero object

(1) Every $f : X \to Y$ has fiber and cofiber.

(2) a diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a pushout $\Leftrightarrow$ it is a pullback.

i.e. X is a fiber of $Y \to Z$ $\Leftrightarrow$

Z is a cofiber of $X \to Y$.

<u>Example.</u> Spaces_* is <u>not stable</u>

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \arrow[ul, phantom, "\lrcorner"] \end{tikzcd} \qquad \Omega \Sigma X \not\simeq X$$

<u>Example.</u> Chain_k is <u>stable</u>

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X[1] \arrow[ul, phantom, "\lrcorner"] \end{tikzcd} \qquad X[1][-1] \simeq X$$

<u>Defn.</u> Let $[-1] = \Omega : \mathcal{C} \to \mathcal{C}$

$$X \mapsto \text{fiber}(0 \to X)$$

and

$$[+1] = \Sigma : \mathcal{C} \to \mathcal{C}$$

$$X \mapsto \text{cofiber}(X \to 0)$$

<u>Prop.</u> If $\mathcal{C}$ is stable, then Ω, Σ are inverses. This follows almost by defn, but need to think through some combinatorics.

this is basically the power of ∞-categories.

<u>Prop.</u>

$$\hom(X,Y) \simeq \Omega \hom(X, \Sigma Y) \simeq \Omega^n \hom(X, \Sigma^n Y).$$

<u>Proof.</u>

$$\begin{tikzcd}
X \arrow[dr] \arrow[drrr, bend left] \arrow[ddrr, bend right] & & & \\
& Y \simeq \Omega\Sigma Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

<u>Defn.</u> $\forall X, Y,\ n \in \mathbb{Z}$ define

$$\mathrm{Ext}^n(X,Y) = \pi_{-n} \hom(X,Y)$$

$$:= \pi_0 \hom(X, \Sigma^n Y).$$

<u>Cor.</u> LES of Ext-groups.

<u>Proof</u> (sketch). Given a fiber sequence

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ 0 \arrow[r] & X_2 \end{tikzcd}$$

$\hom(-,Y)$ $\rightsquigarrow$

Fibration

$$\begin{tikzcd} \hom(X_0,Y) & \hom(X_1,Y) \arrow[l] \\ 0 \arrow[u] & \hom(X_2,Y) \arrow[l] \arrow[u] \end{tikzcd}$$

this is a limit (fiber). But any fibration gives LES in π_n and this is the LES of Ext groups.

$\boxtimes$

$\overset{=}{30?}$

<u>Prop.</u> The homotopy category of a stable ∞-cat $\mathcal{C}$ is additive.

<u>Proof.</u> We've already seen

$$\pi_0 \hom(X,Y) = \pi_0 \Omega^n(-,-) = \pi_n(-,-)$$

is an abelian group. Observe

$$X \oplus Y \simeq \mathrm{cofiber}(X[-1] \xrightarrow{0} Y)$$

∎

<u>Prop.</u> A stable category $\mathcal{C}$ admits all finite limits and colimits.

<u>Proof.</u> Enough to show (co)equalizers + (co)product exist. But

$$\mathrm{colim}(X \underset{g}{\overset{f}{\rightrightarrows}} Y) = (X \xrightarrow{g-f} Y)$$

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