280x (19.11)

Homological Mirror Symmetry - Intro (Magic School Bus!)

X alg. var. smooth compact. We study $D^b\,Coh(X)$. Goal of Mirror Symmetry is to find $X^\vee$ symplectic variety and $D^\pi\,Fuk(X^\vee)$

Defn Y^{2n} even dim mfld. If there exists ω 2-form: $d\omega = 0$, non-degenerate:

$$T_p X \xrightarrow{\sim} T_p^* X$$
$$v \longmapsto \omega(v, -)$$

then we say Y is symplectic.

Example. $(\mathbb{R}^2, \omega = dx \wedge dy)$.

Example. $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$

Example. Take N any smooth mfld. Then T^*N the cotangent bundle is symplectic.

In coordinates: p_i, q_i set $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$

on the mfld — on the cotangent

Originally, symplectic topology appeared in classical mechanics: take particle in N. Its position + momentum are given by point in $Y = T^*N$. This is called the <u>phase space</u> of the system.

Take $H : T^*N \to \mathbb{R}$ "energy" (or Hamiltonian of the system). Can use (H, ω) to define time evolution.

use the pairing to get vector field X_H

<u>Def.</u> $dH \xrightarrow{\omega} X_H \in \Gamma(Y, TY)$

Integrate the flow along X_H to get $\phi_H(t) : Y \to Y$ the time evolution. Now ϕ_H is a symplectomorphism (i.e. $\phi_t^* \omega = \omega$).

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SOFT

(1) A general symplectic mfld Y has ∞-dim space of symplectomorphism (contrast with f.d. automorphism group of alg. var.)

(2) All Y are locally symplectomorphic:

If $p \in Y$, $p' \in Y'$. Then $\exists$ nbhd $p \in \mathcal{U}$, $p' \in \mathcal{U}'$ such that $\mathcal{U} \xrightarrow[\sim]{\text{symp}} \mathcal{U}'$

<u>Defn.</u> Lagrangian $\mathcal{L}^n \in Y^{2n}$ is a submfld such that $\omega|_{\mathcal{L}} = 0$.

Going back to our examples and finding the Lagrangian...

<u>Example.</u> $(\mathbb{R}^2, dx \wedge dy)$ — every curve.

<u>Example.</u> $(\mathbb{R}^{2n}, dx \wedge dy)$ - for example $\{x_i = 0\}$ and many more.

<u>Example.</u> T^*N any conormal bundle to subvariety, for example the zero section.

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This example is actually universal:

3. $L \subset Y$ Lagrangian, then $\exists\, \mathcal{U} \subset L$ tubular nbhd s.t. $\mathcal{U} \cong T^*L$.

<u>Example.</u> X Kähler space, $\omega(v,w) = \mathrm{Im}(\quad)$

symplectic form on Kahler space

the Kahler

So we want to convince you Lagrangian are kind of the basic building blocks of symplectic topology. For example,

Y, Y' symplectic $\Rightarrow Y \times Y'$ symplectic.

$f: Y \to Y'$ diffeo $\Rightarrow \Gamma_f \subset Y \times Y'$.

Then

$$\boxed{f \text{ symp} \Leftrightarrow \Gamma_f \text{ Lagrangian}}$$

Should actually consider time dependent to get intersection points

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Question. Given Y symp, $H : Y \to \mathbb{R}$

$\phi_H(t) : Y \xrightarrow{\sim} Y$. Want Time 1

periodic orbits on Y. Then

$$\Gamma_{\phi_H(1)} \subset Y \times \overline{Y}$$

$\Gamma \cap \Delta$

graph diagonal

Conclusion. Intersection points of Lag are very interesting.

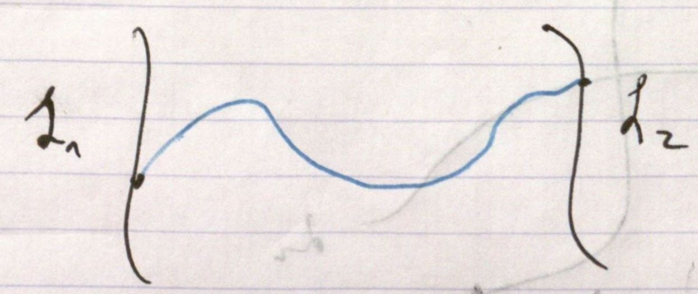
Want: Alg. packaging of Lagrangian + intersections. We'll try some approximations...

For simplicity sake, assume Y has complex structure.

0.1 Stupid(Y) ob = Lagrangian

$\mathrm{Hom}(L_1, L_2)$ = "path of $L_1 \to L_2$"

$= P(L_1, L_2)$



(0.2) Would like to have

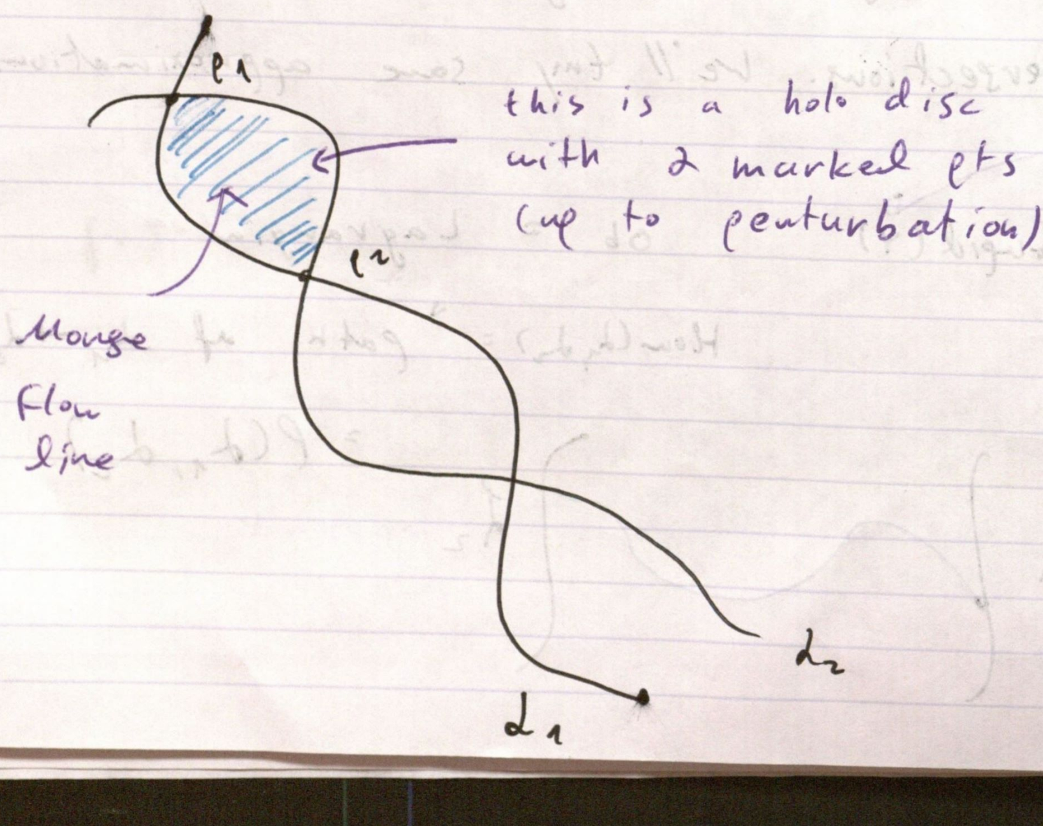
$$\mu_F : \mathcal{P}(\alpha_1, \alpha_2) \longrightarrow \mathbb{R},$$

and $\mathrm{Hom}(\alpha_1, \alpha_2) = (C^{\bullet}_{\mu}(\alpha_1, \alpha_2), d_\mu)$

why is that reasonable?

$\mathrm{Crit}(\mu_F) = \alpha_1 \cap \alpha_2$ with $\mu_F \sim$ length

$$d_\mu : \underbrace{\bigoplus_{p \in L_1 \cap L_2} \mathbb{k}\langle p\rangle}_{C^{\bullet}_{\mu}(\alpha_1, \alpha_2)} \longrightarrow \bigoplus_{p \in L_1 \cap L_2} \mathbb{k}\langle p\rangle$$



(want to have < ∞ space of discs)

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So d_p counts holomorphic disc w/ sign.

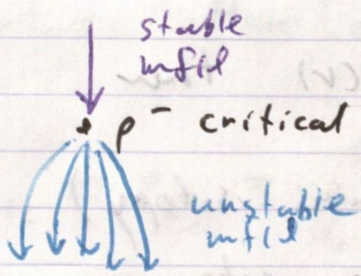
Let's stop for a second and give a 5 sec intro to Morse theory.

critical point generic, non-degenerate

Given M, $h: M \to \mathbb{R}$, the Morse complex

$$= \bigoplus_{p \in Crit(h)} \mathbb{k}\langle p \rangle, \quad \mathbb{k} \text{ coeff field}$$

Dp



p - critical point of h

$\deg(p) =$ dim unstable mfld of p.

The differential $d: p \mapsto \sum_{\substack{\deg(q) \\ = \deg(p)-1}} q \cdot c_{p,q}$ count flow lines

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When you leave the class please forget this pictures ... they are not accurate and meant as motivation ... ☺

<u>Stupid ~~idea~~ 2</u>

ob $\mathcal{L}$ lagrangian

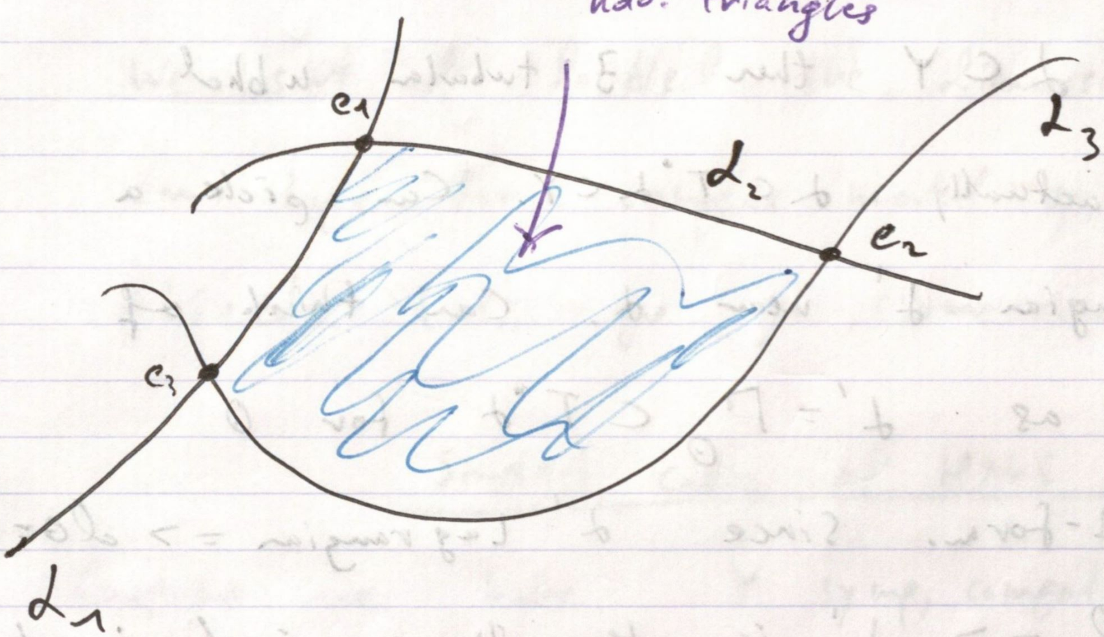
Mor $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = H^0(C_F(\mathcal{L}_1, \mathcal{L}_2))$

<u>Now we want</u> if $\exists H(r)$ then

$\phi_H(t)(\mathcal{L}_1) = \mathcal{L}_2$ (Hamitonian isotopy)

then $\mathcal{L}_1 \simeq \mathcal{L}_2$ inside our category.

<u>Note.</u> this more are less follow from how composition of morphism is defined. In terms of the picture



Want $\mathrm{Hom}(\alpha_1, \alpha_2) \otimes \mathrm{Hom}(\alpha_2, \alpha_3) \to \mathrm{Hom}(\alpha_1, \alpha_3)$

$$e_{12} \otimes e_{23} = \sum c_{123} \cdot e_{13}$$

counts of hol. triangles

Pick $\mathcal{L} \subset Y$ then $\exists$ tubular nbhd so actually $\mathcal{L} \subset T^*\mathcal{L} \subset Y$. Can pick a Lagrangian $\mathcal{L}'$ near $\mathcal{L}$. Can think of this as $\mathcal{L}' = \Gamma_{\Theta} \subset T^*\mathcal{L}$ for Θ a 1-form. Since $\mathcal{L}'$ Lagrangian $\Rightarrow d\Theta = 0$. If $\Theta = df \Rightarrow \mathcal{L}'$ is Hamiltonian isotopic to $\mathcal{L}$.

So $H^1(\mathcal{L}, \mathbb{R})$ is an upper bound on how many Lagrangians we have.

Let's complexify!

(0.3) Pick a flat $U(1)$ connection ξ

this is still a Lie...

Fuk is really complicated...

on $\mathcal{L}$. Call $(\mathcal{L}, \xi)$ object of the Fuk(X).

but it will do for this talk.

Now

$$\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = H^*(C^*_{\mathrm{F}}(\mathcal{L}_1, \mathcal{L}_2), d_{\mathrm{F}})$$

$$\mathrel{\|}$$

$$\bigoplus_{p \in \mathcal{L}_1 \cap \mathcal{L}_2} \mathbb{k}\langle p \rangle \otimes \mathrm{Hom}(\xi_1, \xi_2)_p$$

We sort of doubled the dimension...

can hope for complex moduli space of objects. Let's go to some examples.

Simplest case of HMS

Suppose we have Y symp, compatible cx structure and Ω hol. vol. form.

Defn. L is special Lagrangian if

$$\Omega|_L = e^{i\theta} \cdot \Omega_{\mathbb{R}}, \quad \theta \text{ constant.}$$

Suppose Y contains special Lagrangian torus $(S^1)^n = L_1 \subset Y$. Then the statement we made before is true for special Lagrangians i.e. have $H^1(L_1, \mathbb{R})$ - dim family of special Lagrangian tori.

Best case scenario. Y is special Lagrangian bundle over B^n.

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Let X - dual bundle. $X \leftarrow L_0^\vee$
$\downarrow$
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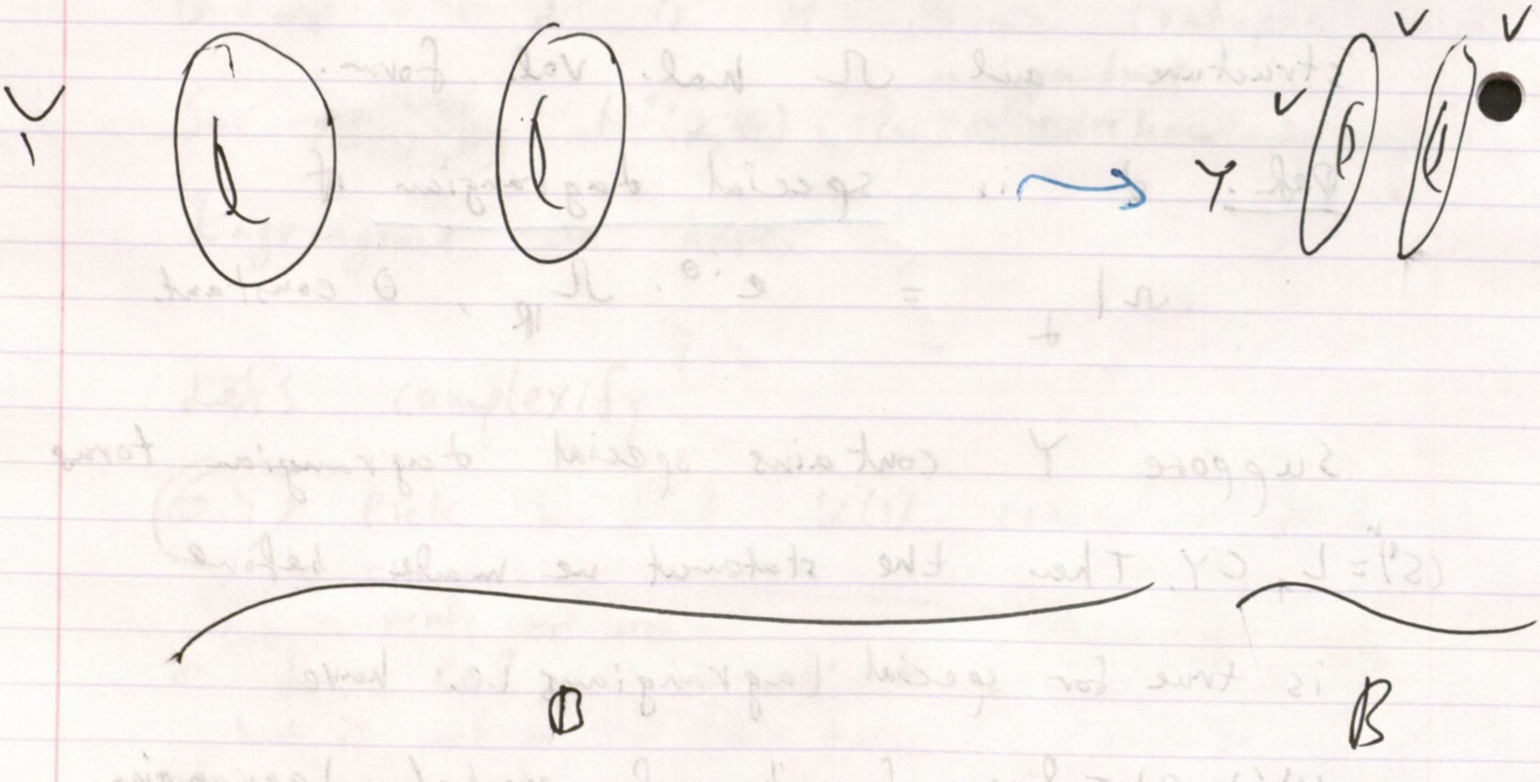
$L_0^\vee = Hom(\pi_1(L_0), U(1))$. Picture



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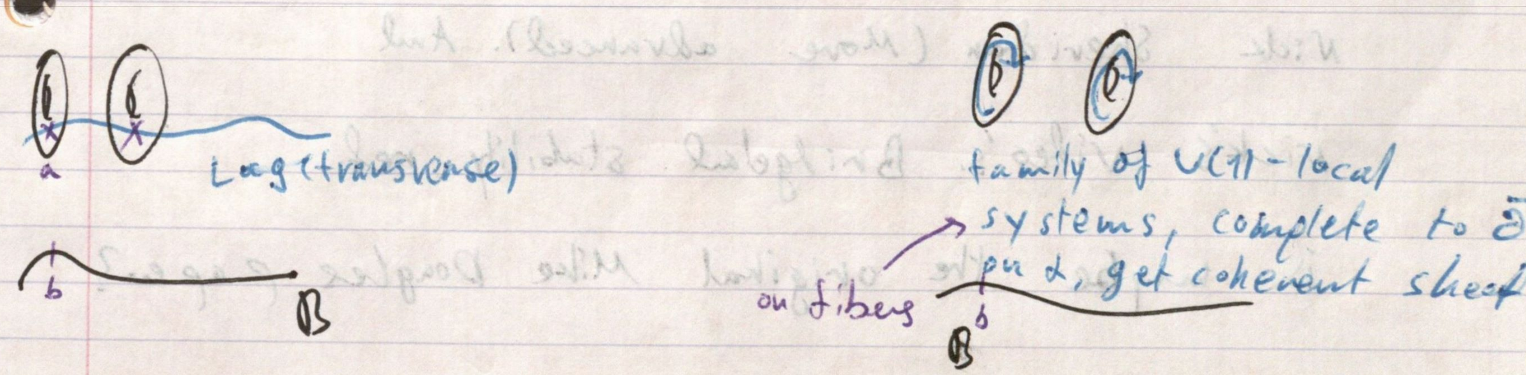
<u>Claim:</u> X cplx CY.

$$D^b \mathrm{Coh}(X) = D^\pi \mathrm{Fuk}(Y)$$

How??? 5-sec idea



Kind of Fourier-Mukai transform.

So cplx structure on Y defines Bridgeland stability on Fuk(Y) (in good cases). Then (α, ξ) is stable (hopefully!) implies we can find a special Lagrangian representative, i.e. find H_t such that $\phi_{H_t}(1)$ sends α to special Lagrangian.

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Ref. First example, is Polishuk-Zaslow the mirror symmetry for elliptic curves.

Ref. More exact Fukaya category paper is Denis Auroux intro paper.

Ref. Kontsevich ICM address.

Also HMS for CY hypersurface by Nick Sheridan (more advanced). And Nick's video's. Bridgeland stability ref is maybe the original Mike Douglas paper?