ZPOX (21.11)

Wall crossing. developed by Joyce-Song, Kontsevich-Soibelman.

Rough idea. Study moduli space of semi-stable obj (or inv related to them).

for an abelian (triangulated) category $\mathcal{A}$ with stability condition Z. Want to know how it changes as Z passes through walls in $\mathrm{Stab}(\mathcal{A})$.

Today we'll only study abelian cat.

Fix Q finite acyclic quiver

$$\mathcal{A} = \mathrm{Rep}^{f.d.}_{\mathbb{C}}(Q)$$

Recall. $K(\mathcal{A}) \simeq \mathbb{Z}^{Q_0}$

Notation. $K(\mathcal{A})^{\geq 0} = \{ [M] \mid M \in \mathcal{A} \} \simeq \mathbb{N}^{Q_0}$ the positive cone.

$\overset{\sim}{30}2$

Thm. for every $\vec{z} = (z_i)_{i \in Q_0} \in (\tilde{\mathbb{H}} \mid \mathbb{R}_{\geq 0})^{Q_0}$,

the following is a stability condition

$$Z(M) := \sum_{i \in Q_0} z_i \dim_{\mathbb{C}} M_i$$

But first we need to talk about

"Moduli Space" of objects of $\mathcal{A}$

For a dimension type $\gamma \in K^{\geq 0}(\mathcal{A})$, write

$$R_\gamma = \bigoplus_{\alpha : i \to j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j})$$

for the representation of Q associated to γ. There is an action on R_γ by

$$GL_\gamma(\mathbb{C}) = \prod_{i \in Q_0} GL_{\gamma_i}(\mathbb{C})$$

One way to define the moduli space

$$\mathcal{M}_\gamma = R_\gamma / GL_\gamma(\mathbb{C})$$

moduli stack of Q-rep with dim type γ

$\stackrel{=}{3}02$

Look at constructible functions on $\mathcal{M}_\gamma$.

Defn. A <u>stratification</u> of a variety X is a decomposition into locally closed subvarieties.

Defn. A <u>constructible function</u> $f: X \to \mathbb{C}$ is a function that can be written

$$f = \sum_{i=1}^{k} a_i \, 1_{X_i}$$

complex numbers

char. function of locally closed $X_i \subset X$.

In other words

$$\boxed{\text{constructible functions are built out of char. functions of strata}}$$

Defn. A constructible function on $\mathcal{M}_\gamma$ is a constructible function on R_γ invariant under $GL_\gamma(\mathbb{C})$.

Some motivation to why we want to consider such functions is

<u>Important Example.</u>

$$\delta_\gamma(M) = \begin{cases} 1 & \text{if } M \text{ is semistable} \\ 0 & \text{if not} \end{cases}$$

where $\mathcal{M} = \coprod_{\gamma \in K(A)^{\geq 0}} \mathcal{M}_\gamma$ the full moduli space.

(<u>Lemma.</u> for fixed Z, semistability is an open condition.)

<u>Def.</u> A <u>HN dimension type</u> is a tuple $\gamma^\bullet = (\gamma^1, -, \gamma^s)$ of elements in the Grothendieck group of the cone $K(A)^{\geq 0}$ s.t. $\phi(\gamma^1) > \_\_ > \phi(\gamma^s)$ (ϕ phase of Z)

For fixed HN type $\gamma^\bullet$, can look at $R^{HN}_{\gamma^\bullet} \subseteq R_{\gamma^1 + \_\_ + \gamma^s}$ consisting of objects having HN filtration of type $\gamma^\bullet$.

Our claim,

Prop: These form a stratification of $\mathcal{M}_\gamma$.

It would follow as conclusion that S_γ are constructible.

Let's write $H_\gamma(\mathcal{A}) = \{$ constructible functions on $\mathcal{M}_\gamma\}$ with $H(\mathcal{A}) = \bigoplus_{\gamma \in K(\mathcal{A})^{\geq 0}} H_\gamma(\mathcal{A})$

Our goal is to put a multiplications on them. It basically should be some kind of convolution of this that "knows about extensions". This is going to be the Hall algebra. But first we'll define another structure that does give an algebra but not the one we want

First Guess $(f \circ g)(M) = \sum f(M_0)\, g(M_1)$

$\{0 \to M_0 \to M \to M_1 \to 0\}/\sim$

SES up to iso of SES

convolution w.r.t. extensions in $\mathcal{A}$.

We need a more sophisticated product. We're gonna use something first done by MacPherson: "integration" of constructible functions! Let $f = \sum_{i=1}^{k} a_i 1_{X_i}$, the integral w.r.t. Euler characteristic

$$\int_X f \, d\chi = \sum_{i=1}^{k} a_i \chi(X_i)$$

χ top. Euler characteristic of X in analytic topology

Define $(f * g)(M) = \int_{M_0 \subseteq M} f(M_0) \, g(M/M_0) \, d\chi$

~~This takes more structure into account,~~

Note. This takes into account some of the alg. structure of representation varieties.

Thm (Kapranov–Vasserot, Joyce) This defines an associative algebra with unit 1_0.

We'll call this the Ringel–Hall algebra.

Easy to prove. Can write

$$(f_1 * \_\_\_ * f_n)(M) = \int_{Flag^n(M)} f_1(M_1/M_0) f_2(M_2/M_1)$$

$$\_\_\_ \quad f_n(M_n/M_{n-1})\, d\chi$$

where $Flag^n(M) = \{ 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \}$.

NB. $H(A)$ is graded over $K(A)^{\geq 0}$,

If M_γ has positive dimension, then $H_\gamma(A)$ is huge. Want more tractable subalg.

Defn. $K_\gamma(M) = \begin{cases} 1 & \text{, if } [M] = \gamma \\ 0 & \text{, otherwise} \end{cases}$

Define,

$$C(A) = \{ K_\gamma \mid \gamma \in K(A)^{\geq 0} \}$$

claim. This is a bialgebra in a natural way.

$$\Delta : H(A) \longrightarrow H(A \times A)$$

$$\Delta f(M, N) = f(M \oplus N)$$

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Note that
$$\begin{array}{c} H(A \times A) \\ \cup \\ H(A) \otimes H(A) \end{array}$$

Thm (Joyce) Δ restricts to a coproduct

$$\Delta : C(A) \to C(A) \otimes C(A)$$

and $(C(A), *, 1_0, \Delta, \eta)$ is a bialgebra (cocommutative). The counit η is evaluation at 0.

Some formulas

Defn. For any pair $\beta, \gamma \in K(A)^{>0}$, not proportional over $\mathbb{Q}$, define the wall

$$W_{\beta\gamma} = \{ Z \in \mathrm{Stab}(A) \mid \phi(\beta) = \phi(\gamma) \}$$

Fixed $\alpha \in K(A)^{>0}$; Collection of semistable objects of class α is locally constant on complement of $\bigcup_{\beta+\gamma=\alpha} W_{\beta\gamma}$

Prop. Fix Z.

$$K_\gamma = \sum_{n\geq 1} \sum_{\substack{\gamma_1+\dots+\gamma_n=\gamma \\ \phi(\gamma_1)>\dots>\phi(\gamma_n)}} \delta_{\gamma_1} * \_\_\_ * \delta_{\gamma_n}$$

Proof. Direct from existence and uniqueness of HN filtrations. ∎

Passing through walls, RHS changes.

Thm (Reineke Inversion formula)

$$\delta_\gamma = \sum_{n\geq 1} \sum_{\substack{\gamma_1+\dots+\gamma_n=\gamma \\ \text{s.t. } \phi(\gamma_1+\dots+\gamma_i)>\phi(\gamma) \\ \forall\ 1\leq i\leq n-1}} (-1)^{n-1} K_{\gamma_1} * \_\_\_ * K_{\gamma_n}$$

Simple minded example.

Our quiver

$$\begin{tikzcd} \underset{v_1}{\bullet} \arrow[r] & \underset{v_2}{\bullet} \end{tikzcd}$$

(we've already done this by hand with Murad)

look at $\begin{cases} \mathbb{C} \xrightarrow{0} \mathbb{C} \\ \mathbb{C} \xrightarrow{\lambda} \mathbb{C}, \lambda\neq 0 \end{cases}$ rep'n of class v_1+v_2

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$$K_{v_1} * K_{v_2} = \begin{cases} 1 & \text{at } [\mathbb{C} \xrightarrow{0} \mathbb{C}] \\ 0 & \text{at } [\mathbb{C} \xrightarrow{1} \mathbb{C}] \end{cases}$$

(Ask

can I form extension

$$(\mathbb{C} \to 0) \to (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to 0) \; !)$$

The other way around

$$K_{v_2} * K_{v_1} = \begin{cases} 1 & \text{at } [\mathbb{C} \xrightarrow{0} \mathbb{C}] \\ 1 & \text{at } [\mathbb{C} \xrightarrow{1} \mathbb{C}] \end{cases}$$

(Recall the formula

$$f_1 * \ldots * f_n(M) = \int_{Flag^n(M)} f_1(M_1/M_0) \ldots f_n(M_n/M_{n-1})\, d\chi$$

$$Flag^n(M) = \{ 0 = M_0 \subset M_1 \subseteq \ldots \subseteq M_n = M \}$$

and $\int_X f \, d\chi = \sum_{i=1}^{k} a_i \chi(X_i)$,

where $f = \sum_{i=1}^{k} a_i 1_{X_i}$)

$\overset{=}{3}0\,2$

Let first do the $\underline{\text{case} \quad \phi(\gamma_1) > \phi(\gamma_2):}$

$$\delta_{\gamma_1+\gamma_2} = K_{\gamma_1+\gamma_2} - K_{\gamma_1} * K_{\gamma_2} =$$

$$\begin{cases} 0 & \text{at} \quad [\mathbb{C} \xrightarrow{0} \mathbb{C}] \\ 1 & \text{at} \quad [\mathbb{C} \xrightarrow{1} \mathbb{C}] \end{cases}$$

Immediate conclusion is that $\mathbb{C} \xrightarrow{1} \mathbb{C}$ is a semistable object.

$\underline{\text{case} \quad \phi(\gamma_1) < \phi(\gamma_2):}$ Apply again Reineke formula

$$\delta_{\gamma_1+\gamma_2} = K_{\gamma_1+\gamma_2} - K_{\gamma_2} * K_{\gamma_1} = 0$$

$\Rightarrow$ no semistable object of class $\gamma_1+\gamma_2$. ∎

<u>Cor.</u> $\delta_\gamma \in C(\mathcal{A})$, which is useful to know and not completely obvious.

Claim. $C(\mathcal{A})$ is a Hopf algebra.

Defn. $f \in C(\mathcal{A})$ is primitive if $\Delta f = 1 \otimes f + f \otimes 1$.

Fact. Primitive closed under $[\,,\,]$. So form a Lie algebra.

Lemma. The primitive inside $C(\mathcal{A})$ are exactly functions supported at the indecomposable

$\rightsquigarrow$ Lie algebra $n(\mathcal{A})$ (sometime called Ringel-Hall algebra). This is graded

$$n(\mathcal{A}) = \bigoplus_{\gamma \in K(\mathcal{A})^{\geq 0}} n_\gamma(\mathcal{A})$$

Lemma. $C(\mathcal{A}) \simeq \mathcal{U}(n(\mathcal{A}))$

Example. Q is a Dynkin diagram. Then we can form the associated simple Lie algebra can decompose $\mathfrak{g} \cong n_- \oplus \mathfrak{h} \oplus n_+$

Thm (Ringel, Schofield) $n(\mathcal{A}) \cong n_+$.

In our example

$\circ \longrightarrow \circ$ is Dynkin for sl_3

so $n_+ = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Dimension

works out (3 indecomposables), indeed:

$$\mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C}$$

$$\mathbb{C} \xrightarrow{1} \mathbb{C}$$