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The Minimal Model Program and Stability Conditions (Matthew Woolf)

What is the MMP? Turns birational geometry into biregular geometry. (Given two varieties with the same function field, how are they related explicitly?)

Idea: Find a "canonical" representative in each birational class.

Maximal doesn't work because can always blow up. ⇝ Find a minimal representative.

Minimal: get it by contracting subvarieties.

Given $\pi : X \overset{\text{birat}}{\dashrightarrow} \underset{\text{proj}}{Y}$ $\exists$ very ample line bundle L on Y.

$\pi^* L$ is a basepoint-free line bundle on X, the map to $\mathbb{P}^n$ factors through π.

X, L' a line bundle, $\sigma_0, \dots, \sigma_n \in H^1(L')$, $x \mapsto [\sigma_0(x), \dots, \sigma_n(x)]$

We want to study basepoint-free (free) line bundles. Take really large multiples ⇝ connected fibres

If X is a variety, the Neron-Severi group $NS(X) = \{\text{line bundles}\} / \underbrace{\text{numerical equiv}}_{\substack{\text{Same intersection numbers}\\ \text{with all curves}}}$ is a finitely generated abelian group.

$$H^1(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}) \cong NS(X)$$

$$N^1(X) = NS(X) \otimes \mathbb{R}$$

Nef cone $\mathrm{Nef}(X) = \{ D : D \cdot C \geq 0 \ \forall C \}$

Thm $\mathrm{Nef}(X)^\circ = \mathrm{Amp}(X)$

Free divisors are nef.

Hope: every nef divisor is semiample (i.e. some multiple is free).

Say L is free. L defines a map $f : X \to Y$, Y normal.

Exceptional locus $\mathrm{Exc}(f) = \{ x : f \text{ is not an isomorphism at } x \}$.

↳ codim
- $= 0$ maps to a lower-dimensional variety
- $= 1$ divisorial contraction
- > 1 singular space (small contraction)

Want varieties to be $\mathbb{Q}$-factorial (every Weil divisor has a Cartier multiple). Then can define intersection numbers of Weil divisors with curves (may be rational number).

Given a line bundle L, $R(L) = \bigoplus H^0(L^{\otimes i})$. If this is finitely generated, can form $\mathrm{Proj}\, R(L)$. If L is semistable, this gives stable map to it.

for $X \to Y$, a curve C is contracted iff $L(C) = 0$.

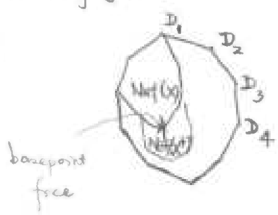
Small contraction $\rho(X) = \rho(Y) + 1$ $\exists D$ such that $D \cdot C < 0$ for contracted.

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{birat}", "\text{isom}"'] \arrow[dr, "\pi"'] & & X^+ \arrow[dl] \\ & Y & \end{tikzcd} \qquad \tilde{D} \cdot C > 0 \qquad X^+ = \mathrm{Proj}\left(\bigoplus \pi_*(iD)\right)$$

exceptional locus has codim ≥ 2

$R(X; D_1, \dots, D_n) := \bigoplus_{i_1, \dots, i_n} H^0(\otimes\, i_j D_j)$

Suppose it is finitely generated.



<u>Conj</u> If $D_i = K_X + B_i$, B_i nice, then the ring is finitely generated.

<u>Stability Conditions</u>

Given stability condition σ and $c \in K(X)$, we get $M_\sigma(c)$ = moduli space of S-equivalence classes of σ-semistable objects of class c.

E, E' are S-equivalent if they have the same Jordan-Hölder factors.

Assume heart is constant

wall

$\sigma_{<0}$ $\quad \sigma_0 \quad$ $\sigma_{>0}$ $\quad \sigma_t$

$\exists$ SES $\quad 0 \to A \to B \to C \to 0$

$\sigma_{<0}(A) < \sigma_{<0}(B)$

$\sigma_0(A) = \sigma_0(B)$

$\sigma_{>0}(A) > \sigma_{>0}(B)$

B stable	B strictly semistable	B unstable
$\sigma_{<0}$	σ_0	$\sigma_{>0}$
M_-	M_0	M_+

$$\begin{tikzcd} M_- \arrow[dr, "\pi_-"'] & & M_+ \arrow[dl, "\pi_+"] \\ & M_0 & \end{tikzcd}$$

$\pi_-^{-1}([B]) \cong \mathbb{P}\mathrm{Ext}^1(C, A)$

$\pi_+^{-1}([B]) \cong \mathbb{P}\mathrm{Ext}^1(A, C)$

<u>Thm</u> (Bayer-Macrì) There is a natural nef class $\mathrm{Nef}(M_\sigma(c))$ which depends continuously on σ.

Caveat: $M_\sigma(c)$ is not known to exist.

$L_\sigma \cdot C = 0$ iff C consists of a single S-equivalence class.

Fact: on surfaces, moduli spaces of sheaves (in particular Hilbert scheme of points) are isomorphic to moduli of Bridgeland-stable objects

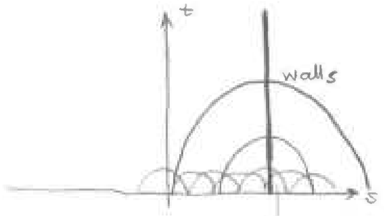
Let's say you have such a space. What are the models which occur in the MMP? Ans: (almost) other moduli spaces of Bridgeland-stable objects.

On $\mathbb{P}^2$, we have $D^b(\mathbb{P}^2) \cong D^b\big(\bullet \overset{x_0, y_0, z_0}{\rightrightarrows} \bullet \overset{x_1, y_1, z_1}{\rightrightarrows} \bullet \mid y_1 x_0 = x_1 y_0$ etc $\big)$ (commutativity)

$a\,\mathcal{O}(k) \longrightarrow b\,\mathcal{O}(k+1) \longrightarrow c\,\mathcal{O}(k+2)$

coefficients of matrix of linear forms of above gives

Geometric stability conditions



$t > 0$

This is a slice of the $GL_2^+(\mathbb{R})$- action.

different values of k

$\forall c\ \forall \sigma, \exists\ \sigma' \in \underbrace{\text{quiver region}}_{\text{in one of the small semidisks at the bottom}}$ such that $M_{\sigma'}(c) \cong M_\sigma(c)$

M_σ can be constructed by GIT.

If $D^b(S) \simeq D^b(S')$, S, S' both K3 , show $\forall\ \sigma \in \mathrm{Stab}\ S,\ c \in K(S)$,

$M_\sigma(c) \cong$ moduli space of (twisted) sheaves on S' for some S'.