Polyn. Stab Cond.

Reconstructing a variety from D^bCoh isn't always possible, but maybe you can w/ D^bCoh + stability condition

But we can't construct stab. conds for most varieties in high dim

Physics wants us to consider

$$Z_{\beta,\omega}(E) = -\int_X e^{-\beta - i\omega} ch(E) \sqrt{td(X)}$$

- unknown how to construct a t-str s.t. $Z_{\beta+i\omega}(E) \in \mathbb{H}_{\geq 0}$ $\forall E \in \heartsuit$.

Observe: If support of a qcoh sheaf E is d,

$$Z_{\beta,m,\omega}(E[\lfloor \tfrac{d}{2} \rfloor]) \to i\infty \text{ or } -\infty.$$

as $m \to \infty$. (Dependency on m later.)

Vague goal: - Define t-structures

- Define phase as we go to ∞.

① Defn of PBSC.

Let X be normal + projective.

Idea: Number $Z(E) \in \mathbb{C}$ ⇝ Polynomial $Z(E) \in \mathbb{C}[m]$

phase $\phi(E) \in \mathbb{R}$ ⇝ phase factor $\phi(E)(m)$

$\phi(E) \in [0,1)$ ⇝ $\phi(E)(t)$
$\uparrow$
$(\phi_0(E)(t),$
$\phi_0(E)(t)+1)$
for some ϕ_0, and
$t \gg 0$

Defn A poly phase fxn is a continuous map $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ s.t.

$\exists f \in \mathbb{C}[m]$ with
$$f(t) \in \mathbb{R}_{>0} e^{\pi i \phi(t)}$$
$(\forall \ t \in \mathbb{R})$

where we identify two ϕ if they agree eventually.

Exer $\phi < \psi \iff \phi(m) < \psi(m) \ \forall \ m \gg 0$ defines a total order.

Defn A choice of polynomial stability function (PSF) on an abelian category A is a group homomorphism

$$K(A) \longrightarrow \mathbb{C}[m]$$

s.t. $\exists$ φ_0 Poly phase fn s.t. $\forall E$, we can choose $\varphi(E)$ PPF s.t. $\varphi(E) \in [\varphi_0, \varphi_0 + 1)$.

↑ in terms of order relation above

Generalise H-N filtration.

Defn A poly stab cond (PSC) on a triang. cat. $\mathcal{D}$ is
- a bdd t-structure
- a PSF on $\mathcal{D}^{\heartsuit}$ w/ H-N prop.

How to construct PSCs?

Propn A abelian
$\mathcal{D}$ triang.

If $Z: K(A) \to \mathbb{C}[m]$ is a PSC s.t. there are no infinite chains of subobjects of strictly increasing phase, no inf. chains of quotients of decreasing phase, then Z satisfies H-N prop.

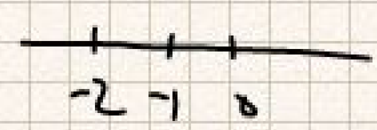
Problem For "standard" PSCs (those motivated by physics) neither hypothesis of Prop'n B satisfied.

② Construction of standard PSCs.

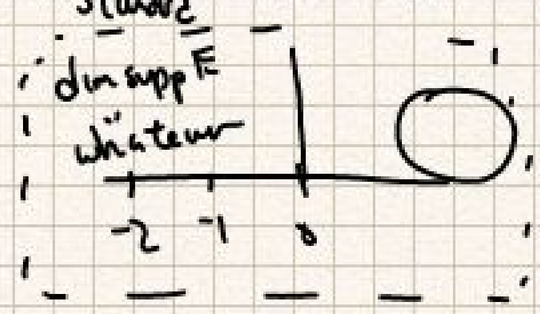
Need more t-structures. We'll introduce perverse sheaves

Ⓐ Perversity.

Standard:



$\mathcal{D}^{\le 0}_{\text{standard}}$ looks like



Set $p(-\infty) = \infty$

$p(d) = 0$ otherwise

Note $\mathcal{D}^{\le 0}_{std} = \Big\{ E \in D^b(X) \text{ s.t. } p(\dim \operatorname{supp} \mathcal{H}^k(E)) \ge -k \Big\}$

Choose other $p : \{0, \dots, \underset{\underset{\dim X}{\uparrow}}{n}\} \longrightarrow \mathbb{Z}$

Thm (Bezrukavnikov, Kashiwara)

If $p(d) \geq p(d+1) \geq p(d) - 1$

(called "perversity fn")

then

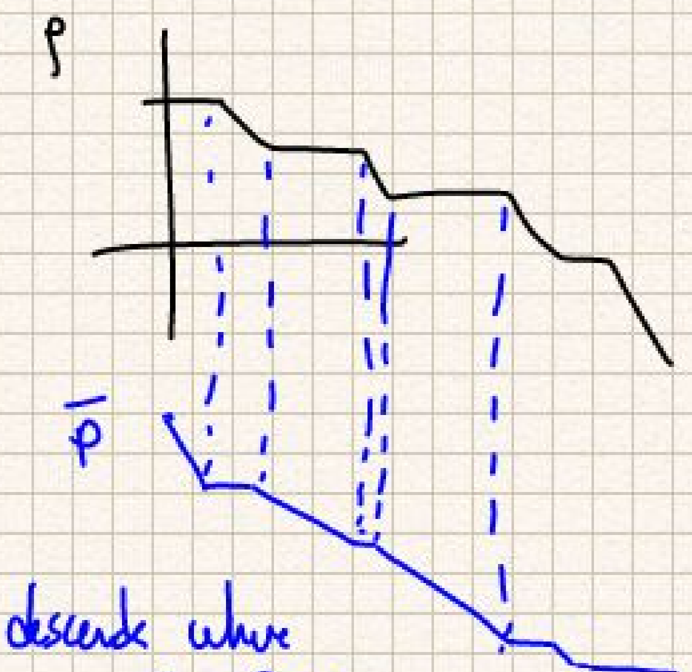
$$D^{p\leq 0}, \; D^{p\geq 0}$$

is a ~~bdd~~ t-structure.

Note If p is a perversity fn,

so is

$$\bar{p}(d) = -d - p(d)$$



descends where
p is flat.

Let ω_X be dualizing sheaf (on X^{smooth})

in degree D. Consider

$$\mathbb{D} : D^b(X) \longrightarrow D(X)$$ ← not nec. in $D^b(X)$ again.

$$E \longmapsto R\mathcal{H}om(E, \omega_X)$$

Set $p^{\cdot} = \bar{p} + D - n$.

Set

$$\mathcal{D}^{p^{\circ}, \geq 0} = \{ E \in \mathbb{D}(D^b(X)) \text{ s.t. } p^{\circ}(\dim \operatorname{supp} \mathcal{H}^k(E)) \geq -k \}$$

Thm (Bezru...)
This defines a t-structure on $\mathbb{D}(D^b(X))$,

w/ $\mathbb{D}(D^{p, \leq 0}_{\geq 0}) = D^{p^{\circ}, \geq 0}_{\leq 0}$.

(flips $\geq$ and $\leq$)

and heart $\mathcal{A}^{p^{\circ}}$.

(Heart $\mathcal{A}^{p}$ = perverse sheaves)

If you have perverse sheaf, look at lowest term

Lemma Let $E \in \mathcal{A}^p$ (i.e., E perverse).

k maximal s.t.

$$\mathcal{H}^{-k}(E) \neq 0.$$

Set $\dim \operatorname{supp} E = \dim \operatorname{supp} \mathcal{H}^{-k}(E) = d$.

Then: • $p(d) = -k$

• $\forall i \neq k$, $\dim \operatorname{supp} \mathcal{H}^{-i}(E) < d$.

Pf $p(d) \geq -k$ is clear.

If $\mathcal{F} \in Coh(X)$ w/ dim supp $\mathcal{F}$ $\geq -(k-1)$, then

$Hom(\mathcal{F}[k], \mathcal{H}^{-k}(E)[k])$

$= Hom(\mathcal{F}[k], E)$

$= 0$ by t-structure.

Taking $\mathcal{F} = \mathcal{H}^{-k}(E)$ gives contradiction

for $i < k$:

$g(\dim supp \mathcal{H}^{-i}) \geq -i > -k = p(\dim supp \mathcal{H}^{-k})$

$\Longrightarrow \dim supp \mathcal{H}^{-i} < \dim supp \mathcal{H}^{-k}$.

Ex $p(d) = -\lfloor \frac{d}{2} \rfloor$, middle perversity

Then $\mathcal{A}^p$ = 2-term complexes w/ $H^{-1}(E)$ torsion free, $H^0(E)$ torsion.

Construction Let $\omega \in A^1(X)_{\mathbb{R}}$ be ample. (Like a Kähler form)

$A^1(X)_{\mathbb{R}} = A^1(X) \otimes \mathbb{R}$

$A^1(X)$ = grp of codim 1 subvarieties of X modulo rational equiv.

Let

- $(P_0, \dots, P_n) \in (\mathbb{C}^\times)^{n+1}$ s.t. $P_i / P_{i+1} \in$ upp half plane
- p perversity fun s.t. $(-1)^{p(d)} P_d \in UHP$.
- $U \in A^*(X) \otimes \mathbb{C}$ of the form $1 + N$

something in positive degree, possibly zero class.

Big Thm ∀

$$\Omega = (\omega, P, \rho, U)$$

as above,

$$Z_\Omega : K(X) \longrightarrow \mathbb{C}[m]$$

$$E \longmapsto \int_X \sum_{d=0}^{n} \left(P_d\, \omega^d\, ch(E) \cdot \underset{\text{like } \sqrt{td_X}}{U}\right) m^d$$

is a poly stab cond.

Easy pf:

Z_Ω is psf w/

$\varphi_0 = \varepsilon$ for $\varepsilon > 0$ small.

$(-1)^{p_d} P_d \in \mathbb{H} e^{i\varepsilon}$ ∀ d.

Fix $E \in \mathcal{A}^p$.

Compute:

$$(ch(E) \cdot U)^{n-d} \leftarrow \text{codim } n-d \text{ part}$$

$$= (-1)^k\, ch^{n-d}(\mathcal{H}^{-k}(E))$$

since ω ample,

$$Z(E)(m) = \underbrace{\int_X \omega^d\, ch^{n-d}\, \mathcal{H}^{-k}(E)}_{\in\, \mathbb{R}_{>0}} \cdot (-1)^k P_d m^d + \text{lower order } m^k \text{ terms}$$

$\in \mathbb{H} e^{i\varepsilon}$ for $m \gg 0$.

Recall

$$ch(E) = \sum (-1)^i\, ch(\mathcal{H}^i(E))$$

If $\mathcal{F} \in Coh(X)$ w/

$\dim \operatorname{supp} \mathcal{F} = m$

then $ch(\mathcal{F})$ lives in small dim.

$i : \operatorname{supp}\mathcal{F} \hookrightarrow X$

thus $\mathcal{F} \cong i_* i^* \mathcal{F}$.

By Groth-Riem-Roch,

$$ch(\mathcal{F}) \in \operatorname{image}(i_* : A_\cdot(\underset{\operatorname{supp}\mathcal{F}}{Z}) \to A(X)).$$

Pf of H-N property
(very pretty)

[Pre 1: Duality]

Given Ω, let

$$\Omega^* = (\omega,\ P^* = -\overline{P},\ \overline{p},\ U^* = (-1)^p \mathrm{ch}(\omega)^{-1}_* \cdot P(\overline{U}))$$

conjugation

parity operator.

$P = (-1)^{n-d}$ on $A_d(X)$

Thm

$Z_{\Omega^*} : K(X) \to \mathbb{C}[[m]]$

is a PSF on $\mathbb{D}(\mathcal{A}^p)$

s.t. $\phi(E) > \phi(F) \iff \phi^*(\mathbb{D}(E)) > \phi^*(\mathbb{D}(F))$

(i.e. $\mathbb{D}$ preserves stability.)

Defn Given Abelian $\mathcal{A}$ and a PSF Z, and an object $E \in \mathcal{A}$, a maximally destabilizing quotient (minimally destabilizing subobject, dually) is a quotient $E \twoheadrightarrow B$ (subobject $C \hookrightarrow E$) s.t. $\forall$ other

$E \twoheadrightarrow B'$
$(C' \hookrightarrow E)$

we have $\phi(B') \geq \phi(B)$ w/ equality iff $\exists$ factoring $E \twoheadrightarrow B \twoheadrightarrow B'$.
$(\phi(C') \leq \phi(C))$ $(C' \hookrightarrow C \hookrightarrow E)$

Let

Ex(d) be statement
"max destab quotient
exists for dimsupp(E) $\leq d$"

Term(d) be statement
"MDQ filtration of objects
E w/ dimsupp(E) $\leq d$
terminate."

Note: Ex + Term $\Rightarrow$ H-N prop.
just as in Bridgeland's proof.

<u>Strategy:</u>

$$\begin{tikzcd} \mathrm{Ex}_0 \arrow[r, "\text{implies}"] & \mathrm{Ex}_1 \\ \mathrm{Term}_0 \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Ex}_1 \arrow[d, "\text{implies}"] \\ \mathrm{Term}_0 \arrow[r] & \mathrm{Term}_1 \end{tikzcd}$$

i.e., (I) $\mathrm{Term}_{d-1} + \mathrm{Ex}_{d-1} \Rightarrow \mathrm{Ex}_d$

(II) $\mathrm{Term}_{d-1} + \mathrm{Ex}_d \Rightarrow \mathrm{Term}_d$.

(I) is hard part,

<u>pf of (I)</u> By duality, enough to show
MDSubs exist.
Pick $E \in \mathcal{A}^p$ w/ dimsupp $E = d$.
If E semistable, done.

Otherwise, pick

$$E_1 \hookrightarrow E_0 \twoheadrightarrow E_0/E_1$$

w/ $\phi(E_1) > \phi(E_0)$.

<u>Claim</u> If

$$C \hookrightarrow E_1$$

is MDSubobject, then

$$C \hookrightarrow E_0$$

is MDS.

<u>Pf</u> Choose

$$\begin{tikzcd} & & C' \arrow[d, hook] \\ C \arrow[r, hook] & E_1 \arrow[r, hook] & E_0 \end{tikzcd}$$

Either $\phi(C') < \phi(C)$ (done)
or $\phi(C) \leq \phi(C')$

$\Rightarrow$ by seesaw,

$$\phi(C) \geq \phi(E_1) \geq \phi(E_0/E_1)$$

$\Rightarrow$ if C', E_0/E_1 is semistable
then $C' \to E_0/E_1$ is zero.

$\Rightarrow$ get lift

$$\begin{tikzcd} & C' \arrow[d] \arrow[dl, dashed] \\ E_1 \arrow[r, hook] & E_0 \end{tikzcd}$$

So NTS: C', E_0/E_1 semistable

We want to show: (a) Every non-s.s. $E \in \mathcal{A}^P$ has a s.s. quotient w/ smaller ϕ.

Thm

(b) MDS can be

tested by semistable.

(a) ⟹ (b) by duality.

Thm $^{(c)}$ the process

$$\begin{tikzcd} & E_n \text{ s.s., done} \\ E_n \arrow[ur] \arrow[r] & \text{Have } E_{n+1} \hookrightarrow E_n \twoheadrightarrow B \end{tikzcd}$$

w/ $\Phi(E_{n+1}) > \Phi(E_n)$

needs to terminate, for us to be successful. We'll prove (a), leaving (c) for further reading.

<u>Pf of (a)</u> (Dually, showing ss sub)

Choose $E' \hookrightarrow E$ w/ $\Phi(E') > \Phi(E)$

and $\dim \operatorname{supp} (E/E')$ maximal

If this terminates, we're done.

Otherwise, $\dim \operatorname{supp} (E^i/E^{i+1})$

must be a constant eventually,

call it $d > 0$.

Out of time. //

Concluding remarks.

For S K3 surface,

$$\beta \in A^1(X)_{\mathbb{R}}$$

$$\omega \in A^1(X)_{\mathbb{R}} \text{ ample}, \quad \omega^2 >> 0.$$

Bridgeland showed

$$Z_{\beta,\omega} = -\int_X e^{-\beta - i\omega} ch(E)\sqrt{td(X)}$$

is a BSC. (Hard part is t-structure.)

We get stab conditions

$$Z_{\beta, m\omega}$$

$\uparrow$ scale ω.

<u>Thm</u> For $p(d) = \lfloor \frac{d}{2} \rfloor$

$$\rho_d = \frac{(-1)^d}{d!}, \quad u = e^{\beta}\sqrt{td\,X}$$

we get Z_Ω.

$E \in D^b(S)$ is $Z_{\beta,m\omega}$-stable for ω large

$\Rightarrow$ E is Z_Ω stable.

$\Rightarrow$ H-N filtration agrees eventually.