Oct 2

Geometric realisation of simplicial sets:

First:

$[0] \mapsto \bullet$

$F : \Delta \longrightarrow \mathrm{Spaces}, \quad [n] \longmapsto \Delta^n$

$[1] \mapsto$ —

In general want something like $\dfrac{\coprod X_n \times \Delta^n}{\{(fx,u) \sim (x,fu),\ f \in \mathrm{Mor}\,\Delta\}}$ ← more concretely (factoring f as face, degeneracy) we identify $(d_i x, u) \sim (x, \partial_i u)$

↑ for sSet $\ni X_\cdot$, we get $|X|$, realization

What is F explicitly? (on morphisms)

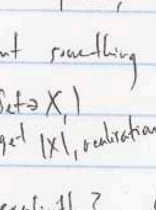
$$\left(\begin{array}{ccc} [n] & \hookrightarrow & [n+1] \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 3 \end{array}\right) \longmapsto \left(\begin{array}{ccc} \Delta^n & \hookrightarrow & \Delta^{n+1} \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^{n+1} \\ x_0 & \mapsto & x_0 \\ x_1 & \mapsto & x_2 \\ x_2 & \mapsto & x_3 \end{array}\right)$$

More generally: $f : [n] \longrightarrow [m]$ induces $Ff : \Delta^n \longrightarrow \Delta^m$ which is linear and sends $x_i \mapsto x_{f(i)}$.

So what does $(fx, u) \sim (x, fu)$ mean? $f : [n] \longrightarrow [m]$, $x \in X_m$, $u \in \Delta^n$; $fx \in X_n$, $fu \in \Delta^m$

E.g. $[n] = [0]$, $[m] = [1]$, $f : 0 \mapsto 1$

$x \in X_1$, $u \in \Delta^0$, $fx \in X_0$, $fu \in \Delta^1$



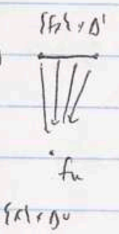
$X_1 \times \Delta^1 \ni \{x\} \times \Delta^1$, $\uparrow fu$, identify

$X_0 \times \Delta^0 \ni (fx, u)$ $\bullet_u$

E.g. $[n] = 1$, $[m] = 0$, $f : \begin{smallmatrix} 0 \mapsto 0 \\ 1 \mapsto 0 \end{smallmatrix}$ $\quad x \in X_0$, $fx \in X_1$, $u \in \Delta^1$, $fu \in \Delta^0$

$X_1 \times \Delta^1 \ni (fx, u)$

$X_0 \times \Delta^0 \ni (x, fu)$



This gives a "degenerate simplex", which comes from surjective maps $f \in \mathrm{Mor}\,\Delta$.

So degenerate simplices don't contribute to topology to geometric realization. But including them let's us say something natural like

$$|X_\cdot| \times |Y_\cdot| \simeq |X_\cdot \times Y_\cdot|$$

which doesn't work if $X_\cdot, Y_\cdot$ are just semisimplicial (i.e. if we don't have degeneracies)

Rem Geometric realization is actually a functor

Also have functor in opposite direction:

$$\mathrm{Sing}: \mathrm{Spaces} \longrightarrow \mathrm{sSet}$$
$$Y \longmapsto X_\bullet, \text{ where } X_n := \mathrm{Hom}(\Delta^n, Y).$$

More generally if we have $G: \Delta \to C$ then
composing with the Yoneda embedding for C gives a new functor $C \to \mathrm{sSet}$.

Next claim: $|-|$ and $\mathrm{Sing}(-)$ are adjoint.

- suppose we have $X_\bullet \to \mathrm{Sing}(Y)$, i.e. $X_n \xrightarrow{\varphi_n} \mathrm{Hom}(\Delta^n, Y)$ compatible ~~with face/degen maps~~
we get a map $|X_\bullet| \to Y$ by sending $(x_n, u) \mapsto \varphi_n(x_n)(u)$

$$|X| = \frac{\coprod X_n \times \Delta^n}{\sim}$$ ✓ satisfies/compatible with $\sim$ by compatibility of φ_n: $\left(\begin{aligned} \varphi(f(x))(u) &= f(\varphi)(x)(u) \\ &= \varphi(x)(f(u)) \end{aligned} \right)$
↑ only nontrivial step

<u>Exercise</u>: check this bijection is natural

<u>Kan Complexes</u>

First define "horns" $\Lambda^n_k := \partial\Delta^n - k^{th}$ face, $0 \le k \le n$

(e.g. ${}_1\overset{0}{\Delta}{}_2 = \Delta^2$, then $\Lambda^2_0 = {}_1\overset{0}{\Lambda}{}_2$
$\Lambda^2_1 = {}_1\overset{0}{\angle}{}_2$)

"Direction of simplices is really important"

<u>Def</u>

A simplicial set K is a <u>Kan complex</u> if $\forall f: \Lambda^n_k \longrightarrow K$ there is an extension $\hat{f}$:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

<u>Thm</u> Let Y a space. Then $\mathrm{Sing}(Y)$ is a Kan complex.

<u>Proof</u> Choose any $f: \Lambda^n_k \to \mathrm{Sing}\, Y$. Use adjunction with geometric realization.
Have diagram

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "|f|"] & Y & & \Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Sing}\, Y \\ {|\Delta^n|} \arrow[u, "d"] \arrow[ur, "|f| \circ d"'] & & \arrow[r, squiggly, "\text{adjunction}"] & \Delta^n \arrow[ur, "\mathrm{Sing}(|f| \circ d)"'] & \end{tikzcd}$$

where d is deformation retract (e.g.  $\to \Lambda$ by "pushing up" from bottom face)