∞-categories Reading Course Notes

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1.1 Geometric realization of simplicial sets

We can define a functor from Δ, the cateogry of simplicial sets, to spaces, object-wise by sending $[n]$ to Δ^n, the standard geometric n-simplex. Morphism-wise, if the original morphism was an order-preserving injection $f : [n] \to [m]$ to an Ff which sends the coordinate x_i in $\Delta^n \subset \mathbb{R}^n$ to the coordinate $x_{f(i)}$ in $\Delta^m \subset \mathbb{R}^m$. Recall that in the geometric realization of a semisimplicial set, we obtained a space via

$$(X_n \times \Delta^n) \,/\, ((d_i x, u) \sim (x, \delta_i u)) \,.$$

In the geometric realization of a simplicial set, our gluings will be obtained not just by identifying elements by the injective maps but also by the degeneracy maps going in the other direction, giving

$$(X_n \times \Delta^n) \,/\, ((fx, u) \sim (x, fu)) \,.$$

Let's look at what the gluing $(fx, u) \sim (x, fu)$ does: $x \in X_m$, $u \in \Delta^n$, and $fx \in X_n$, $fu \in \Delta^m$. We are familiar with what happens when f is order-preserving injective. In the case of f order-preserving surjective: take the example of $[n] = 1$ and $[m] = 0$ so that $x \in X_0$, $u \in \Delta^1$, $fx \in X_1$, and $fu \in \Delta^0$, then the entire 1-simplex gets glued down to the point.

Recall that products and geometric realization don't commute for semi-simplicial sets but they do for simplicial sets.

Let's now define a functor going the other way, from spaces to simplicial sets. We can just take Y at degree n to $\mathrm{Hom}(\Delta^n, Y)$. Note that with the Yoneda embedding, we can actually same similar things after replacing "spaces" with "categories."

1.2 Adjoint Functors

If we have some functors $F : C \to D$ and $G : D \to C$ such that $\mathrm{Hom}(F(X), Y)$ and $\mathrm{Hom}(X, G(Y))$ are in bijection, then we say that they are *adjoint functors*. We expect the following diagram to commute:

$$\begin{array}{ccc}
\mathrm{Hom}(F(X), Y) & \xrightarrow{\circ F(f)} & \mathrm{Hom}(F(X'), Y) \\
\downarrow \Phi_{X,Y} & & \Phi_{X',Y} \downarrow \\
\mathrm{Hom}(X, G(Y)) & \xrightarrow[\circ f]{} & \mathrm{Hom}(X', G(Y))
\end{array}$$

Proposition 1. *Geometric realization and* Sing *are adjoints.*

Proof. Start with a map $\phi : X_\cdot \to \mathrm{Sing}(Y)$, which takes X_n to $\mathrm{Hom}(\Delta^n, Y)$. On the other hand, in $|X_\cdot| \to Y$, take some (x_n, u) on the left and send that to $\phi(x_n)(u) \in Y$. This is well-defined because (fx, u) is sent to $\phi(fx)(u) = f(\phi)(x)(u) = \phi(x)(fu)$. It is left as an exercise to the reader to prove naturality. □

1.3 Kan Complexes

Define the *horn* $\Lambda^n_k = \partial\Delta^n$ minus the face opposite vertex k.

Say we had a map from Λ^n_k to a simplicial set K and that we can include this into the standard n-simplex Δ^n. If there exists a lifting from Δ^n to K for any f, then we say that K is a *Kan* complex.

Theorem 1. $\mathrm{Sing}(Y)$ *is a Kan complex.*

Proof. By adjointness of singular chain complex functor and geometric realization functor, we can associate to any $f : \Lambda^n_k \to \mathrm{Sing}(Y)$ an $f' : |\Lambda^n_k| \to Y$. We have a deformation retract from $|\Delta^n|$ to $|\Lambda^n_k|$, and thus a map from $|\Delta^n| \to Y$ by $f' \circ d \circ i$ which is f'. We thus get a corresponding map from $\Delta^n \to \mathrm{Sing}\, Y$, because by naturality, composing with inclusion first and last is the same. □

Morally, Kan complexes are simplicial sets that behave nice enough to be thought of as spaces.