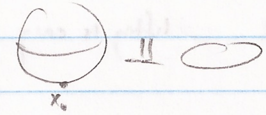
Oct 9

Motivating example

Recall π_1 : $\text{Spaces}_* \to \text{Grps}$. X :  $\pi_1(X, x_0) = 0$, misses other connected components

Fix: fundamental groupoid Π_1 : Spaces $\to$ Grpds

$\rightsquigarrow \Pi_1(X)$ has objects: the pts of X

morphisms: paths between x and y modulo homotopy (obviously everything invertible, hence Groupoid)

Note: $\pi_1(X, x_0) = \text{Hom}_{\Pi_1(X)}(x_0, x_0)$

Rmk Π_1 forgets how paths are homotopic — homotopic paths are equal as morphisms in $\Pi_1(X)$, but can be homotopic in different ways

— i.e., only has 1-dim information

Idea Want Π_2 : Spaces $\to$ "2-Grpds"

Π_2 has objects: pts of X ← has like "morphisms between morphisms"

"1-morphisms": paths x to y (not mod homotopy)

"2-morphisms": homotopy between paths mod homotopies (of maps $[0,1]^2 \to X$)

Rmk Composition is only associative up to homotopy (ie up to a 2-morphism)

Note Π_2 has π_0 (isomorphism classes of objects — isomorphism up to 2-morphism)

π_1 (isomorphism classes of 1-morphisms)

$\pi_2(X, X_0) = \text{Hom}_{\text{Hom}(x_0, x_0)}(x_0, x_0)$

Now we lost ≥ 3-dim'l information — can think of Π_n ...

Want "Π_∞ : Spaces $\to$ ∞-groupoids"

Should be something like — n-morphisms are homotopies of $n-1$ morphisms

Hopefully Π_∞ should classify space up to homotopy equivalence

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Def $\Delta \to \text{Cat}$, $[n] \mapsto [n]$ where category $[n]$ is $\overset{1}{\cdot} \to \overset{2}{\cdot} \to \cdots \to \overset{n+1}{\cdot}$

← unique morphism $i \to j$ $\forall j \geq i$.

clearly morphisms $[m] \to [n]$ in Δ become functors $[m] \to [n]$ in Cat

N : Cat $\to$ sSet is composition of $\Delta \to$ Cat with Yoneda embedding

$N(C)_n := \text{Hom}_{\text{Cat}}([n], C)$ (just like geometric realization)

Horns (E.g.)

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & N(C) \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

lifting is easy -

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] & & C \end{tikzcd}$$

lifts uniquely to

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \cdot f"'] & & C \end{tikzcd}$$

Note however that this doesn't work for Λ^2_0 or Λ^2_2 e.g.

$$\begin{tikzcd} & B \arrow[dr, dashed] & \\ A \arrow[ur, "f"] \arrow[rr, "id_A"'] & & A \end{tikzcd}$$

← lifting is asking for inverse.

Prop (HTT 1.1.2.2) Let K a simplicial set TFAE

(a) $\exists$ category C s.t. $K \simeq N(C)$

(b) $\forall\ 0 < i < n$ and any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a unique lift.

(Similar to Kan complex, but only the inner horns, and need uniqueness)

Sketch (a) $\Rightarrow$ (b) similar to example Λ^2_1 — see book

(b) $\Rightarrow$ (a) Let's just give construction of C.

objects: $K_0 = \{ \Delta^0 \to K \}$

morphisms: $Hom(A,B) = \{ f : \Delta^1 \to K \mid f|_0 = A,\ f|_1 = B \}$

composition:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] & & C \end{tikzcd}$$

lifts uniquely to a 2-simplex τ,

define $g \cdot f = d_1 \tau$

identity $id_A = s_0 A$ = degenerate 1-simplex, satisfies properties by uniqueness of lifting: e.g.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "id_A"] \arrow[rr, "f"'] & s_0 f & B \end{tikzcd}$$

$s_0 f$ is degenerate 2-simplex, boundaries are correct by simplicial identities

associativity exercise (should be the same thing)

Def An ∞-category is a simplicial set K such that $\forall\ 0 < i < n$ any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a lift.

(Motivation: Remember T's shouldn't have unique composition — associativity " up to homotopy)

Rmk By previous proposition, $N(C)$ is an ∞-category. (Categories are special ∞-categories).

Also, $\mathrm{Sing}(X)$ for X a space is an ∞-category! (Kan complex $\Rightarrow$ ∞ category). Also: an ∞-groupoid since we can lift outer horns.

Exercises

1. Fix $[n] \in \Delta$. Let $\Delta^n := \mathrm{Hom}(-, [n]) : \Delta^{op} \to \mathrm{Set}$
   Show $K_n := K([n]) = \mathrm{Hom}_{sSet}(\Delta^n, K)$ for $K \in sSet$.
2. Show $|\Delta^n|$ is homeomorphic to the standard simplex.
3. Let POSets be category with objects posets and morphisms order preserving maps $(p \le q \Rightarrow f(p) \le f(q))$.
   (a) Show the following is a category for any $P \in$ POSets:
   $C(P)$ has objects P,
   $$\mathrm{Hom}_{C(P)}(p,q) = \begin{cases} * & p \le q \\ \emptyset & \text{otherwise} \end{cases}$$
   (b) show this gives a functor $C :$ POSet $\to$ Cat
   (c) what does this have to do with Δ^n?
4. Verify associativity on previous page.