10/2/13

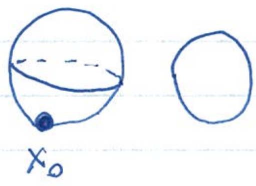
∞-Category Reading Group

Motivational example:

Recall $\pi_1 : \text{Spaces}_* \to \text{Grps}$.

This is inconvenient, e.g. if X is not connected:

$\boxed{X}$



, $\pi_1(X, x_0) = 0$.

Remedy:

$$\Pi_1 : \text{Spaces} \longrightarrow \text{Grpds}$$

$\boxed{\text{categories w/ all morphisms invertible}}$

$$X \longmapsto \Pi_1(X),$$

where $\Pi_1(X)$ has:

obj: set of points of X

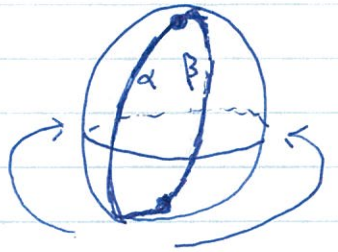
mor: $\hom_{\Pi_1(X)}(x, y) = \left\{ f : [0,1] \to X \;\middle|\; \begin{matrix} f(0) = x, \\ f(1) = y \end{matrix} \right\} \Big/ \sim$

Note: $\pi_1(X, x_0) = \hom_{\Pi_1(X)}(x_0, x_0)$.

Also, $\pi_0(X, x_0) = \{$ isomorphism classes of $\text{ob}(\Pi_1(X))$

Problem: $\Pi_1(X)$ forgets how paths are homotopic.

Ex:



Two different homotopies $\alpha \Rrightarrow \beta$.

This is because $\Pi_1(X)$ disregards 2-dim info.

Better: $\Pi_2(X)$, w/

obj: set of points of X,

mor: $\hom_{\Pi_2(X)}(x,y) = \{f:[0,1] \to X \mid f(0)=x, f(1)=y$

2-mor: $\hom_{\hom(x,y)}(f,g) = \{\text{homotopies } f \Rrightarrow g\}/\sim$

Comp of morphism not associative!

$\boxed{\text{homotopy of homotopies}}$

What is this? Want it to be a functor

$\Pi_2 : \text{Spaces} \longrightarrow$ "2-grpds".

We haven't defined a 2-groupoid, let alone a 2-category!

Note: $\pi_2(X, x_0) = \hom_{\hom(x_0,x_0)}(x_0, x_0)$.

constant path

This is:  .

So $\Pi_2(X)$ has data about dimensions ≤ 2. Could go on to $\Pi_3(X) \in$ "3-grpds," but we'll jump to:

$$\Pi_\infty : \text{Spaces} \longrightarrow \text{"}\infty\text{-grpds"}$$
$$X \longmapsto \Pi_\infty(X),$$

where $\Pi_\infty(X)$ has:

<u>obj</u>: points of X,

<u>mor</u>: paths $f: x \to y$,

<u>2-mor</u>: homotopies $F: f \Rightarrow g$,

$\vdots$

<u>n-mor</u>: homotopies of $(n-1)$-mor,

$\vdots$

This is <u>poetry</u>, not a definition.

<u>Note</u>: Composition is not well-defined, only defined up to higher homotopy.

<u>Hope</u>: $\Pi_\infty(X)$ is a complete invariant.

Categories as sSets:

Last week, Hiro defined $N: Cat \longrightarrow sSet$, the <u>nerve</u>. Recall

$$N(\mathcal{C})_n = \hom_{Cat}([n], \mathcal{C}) = \{ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n \}$$

Facemaps: $0 < i < n$

$$d_i\left(C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} C_n\right) = C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \longrightarrow \cdots \longrightarrow$$

d_0 and d_n delete the morphisms at the ends.

Degeneracies:

$$s_i(C_0 \longrightarrow \cdots \longrightarrow C_n) = C_0 \longrightarrow \cdots \longrightarrow C_i \xrightarrow{id_{C_i}} C_i \longrightarrow \cdots \longrightarrow C_n$$

Can we characterize the image of N? Last week, we saw that for any space X, we have the lifting property:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & Sing(X) \\ \Delta^n_i \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Is there a similar lifting property for $N(\mathcal{C})$?

Consider:

$$\begin{tikzcd}[row sep=small] & c_1 \arrow[r, equal] & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[ur, "f"] & & & c_2 \end{tikzcd}$$

This is a $\Lambda^2_1 \longrightarrow N(\mathcal{C})$. We can easily lift:

$$\begin{tikzcd} & c_1 \arrow[dr, "g"] & \\ c_0 \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & c_2 \end{tikzcd}$$

Moreover, this lift is unique.

__Note__: Would be unreasonable to demand lifting for, e.g., Λ^2_0, since this would mean finding

$$\begin{tikzcd} & c_1 \arrow[dr, dashed, "f^{-1}?"] & \\ c_0 \arrow[ur, "f"] \arrow[rr, "id_{c_0}"'] & & c_0 \end{tikzcd}$$

Not every morphism is invertible.

<u>identities</u>: For $a \in K_0$, define $id_a := s_0 a$.

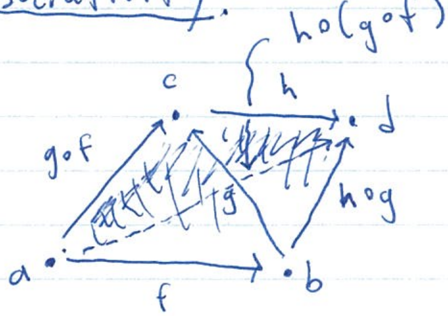
$$a \xrightarrow{\sigma_0 a =: id_a} a,$$

<u>unity</u>:

$$\begin{tikzcd}
& a \arrow[dr, "f"] & \\
a \arrow[ur, "id_A"] \arrow[rr, "f"'] & & b
\end{tikzcd}$$

take to be $\sigma_0 f$.

<u>associativity</u>:



lift the Λ^3_2, opposite face gives

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In light of this proposition, we could've gone back and <u>defined</u> a category to be a simplicial set that satisfies

$$0 < i < n; \quad \begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & K \\
\Delta^n \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd}$$

How do we get ∞-categories? Recall, given

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] & & c \end{tikzcd}$$

in an ∞-category, we want to get <u>a</u> composite, but the composite need not be unique. In particular, any 2-simplex:

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$$

should exhibit has <u>a</u> composite $g \circ f$. Thus we make the following definition:

<u>Def</u>: An <u>∞-category</u> is a simplicial set K such that, for $0 < i < n$, we may lift

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

but <u>not necessarily uniquely</u>!

<u>Note</u>: $N(\mathcal{C})$ is an ∞-category.

<u>Note</u>: $\mathrm{Sing}(X)$ is an ∞-category! Thus, we may define $\Pi_\infty(X) := \mathrm{Sing}(X)$!