Allen Yuan

1) Kan Complexes and Fibrations

Recall the definition of a Kan complex given by David.

Def: $X \in sSet$ is a Kan complex if for any $(n-1)$-simplices $(x_0, \dots, \hat{x}_k, \dots, x_n)$ which are compatible in the sense $d_i x_j = d_{j-1} x_i$, $i<j$, then $\exists\, y \in X_n$ such that $d_i y = x_i$, $i \neq k$.

In other words, every solid diagram (right) extends as shown:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

We can make a relative version of this

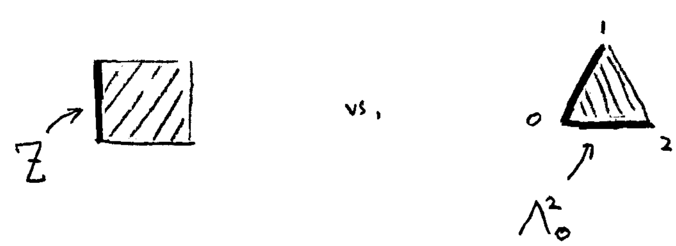
Def: $X, Y \in sSet$. $X \to Y$ is a fibration if every solid diagram extends to the dotted arrow:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Why is this a reasonable definition of fibration?

Our familiar notion of fibration looks like

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Z\times[0,1] \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$



Def: A simplicial set X is a Kan Complex or fibrant if $X \to *$ is a fibration.

Examples: ① $X \in \underline{TOP}$, $Sing(X)$ is a Kan Complex.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Sing(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d] & X \\ {|\Delta^n|} \arrow[ur, dotted] & \end{tikzcd}$$

② Any simplicial group H. (i.e., $H \in \underline{Grp}^{\Delta^{op}}$).
(See other sheet for proof).

③ For any groupoid G, BG is fibrant.
(In particular, for any group G, BG is a Kan Complex).

<u>Pf Sketch:</u>

<u>Observation</u>: $\mathcal{C} \in \underline{Cat}$, $X \in sSet$, then $\mathrm{Hom}(X, B\mathcal{C}) = \mathrm{Hom}(sk_2 X, sk_2 B\mathcal{C})$.

This is because when $X = \Delta^n$, any element of $\mathrm{Hom}(\Delta^n, B\mathcal{C})$ is just a map $[n] \to \mathcal{C}$, which is determined completely by where it takes morphisms and that it preserves commutative triangles. □

Now, in general, $\Lambda^n_k \hookrightarrow \Delta^n$ induces $sk_{n-2}\Lambda^n_k \cong sk_{n-2}\Delta^n$.

<u>Case 1</u>: $n \geq 4$.

$sk_2 \Lambda^n_k = sk_2 \Delta^n$ So the extension
$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & BG \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$
is trivial.

<u>Case 2</u>: $n=3$

$$\begin{tikzcd} \Lambda^3_0 \arrow[r] \arrow[d] & X \\ \Delta^3 \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2\alpha_1"] & \\ 1 \arrow[rr, "\alpha_2"] \arrow[dr, "\beta"'] & & 2 \arrow[dl, "\alpha_3"] \\ & 3 & \end{tikzcd}$$

Observe that if the triangles other than $(1,2,3)$, the commutativity of $(1,2,3)$ is forced.

<u>Case 3</u>: $n=2$. Similar:

$$\begin{tikzcd} & 0 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ 1 \arrow[rr, dashed, "\beta\alpha^{-1}"'] & & 2 \end{tikzcd}$$

□

<u>Observation</u>: Δ^n is <u>not</u> a Kan Complex when $n \geq 2$.
(case 3 fails)

Proof of Moore's Lemma:

$(x_0, \dots, \hat{x}_m, \dots, x_n)$ is compatible in the sense that $d_i x_j = d_{j-1} x_i$ when $i<j$.

$x_i \in H_{n-1}$

Strategy: Build step-by-step some $y \in H_n$ such that $d_i y = x_i$, $i \neq m$.

Step 0: $\cancel{s_0 x_0}$ Since $d_0 s_0 = 1$, $y = s_0 x_0$ will work. ~~for the 0 place~~

Step 1: Consider $s_{n-1}(x_n(d_n y^{-1}))\, y$.

Can check: $d_n\big(s_{n-1}(x_n(d_n y^{-1}))\, y\big)$

$$= (d_n s_{n-1} x_n)\, d_n s_{n-1} d_n y^{-1}\, d_n y$$

$$= x_n \cdot d_n y^{-1}\, d_n y = x_n.$$

$$d_0\big(s_{n-1}(x_n(d_n y^{-1}))\, y\big) = \underline{s_{n-2}\, d_0 x_n}\ \underline{s_{n-2}\, d_{n-1}\, d_0 y^{-1}}\ d_0 y$$

$$= s_{n-2}\big(d_0 x_n\, d_{n-1}\, x_0^{-1}\big)\, d_0 y$$

$$= d_0 y = x_0.$$

Step ℓ: Consider $s_{\ell-2}(x_{\ell-1} d_{\ell-1} y^{-1})\, y$. etc. $\square$

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2) Simplicial Homotopy.

($K, X \in sSet$) $f, g : K \to X$ are simplicial maps.

Def: $f \xrightarrow{\simeq} g$ ("there is a simplicial homotopy from f to g")

if there is a diagram:

$$\begin{tikzcd} K \times \Delta^0 = K \arrow[d, "1\times d^1"'] \arrow[drr, "f"] & & \\ K \times \Delta^1 \arrow[rr, "h"] & & X \\ K \times \Delta^0 = K \arrow[u, "1\times d^0"] \arrow[urr, "g"'] & & \end{tikzcd}$$

h is the homotopy.

As usual, there is a relative version.

Def: $f \xrightarrow{\simeq} g$ (rel L) if $L \subset K$, $f \xrightarrow{\simeq} g$, $f|_L = g|_L$

and
$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "h"] & X \\ L \times \Delta^1 \arrow[u, hook] \arrow[r, "proj_L"'] & L \arrow[u, "\alpha"'] \end{tikzcd}$$
commutes.

In TOP, we could think of homotopies as paths through mapping spaces.

Question: Is sSet enriched over itself? (Answer: Yes.)

If it is, $Hom(X \times Y, Z) = Hom(X, Hom(Y, Z))$.

Letting $X = \Delta^n$, get $Hom(Y, Z)_n = Hom(\Delta^n \times Y, Z)$.

Thus, define $\underline{Hom}(X, Y) \in sSet$ as $[n] \longmapsto Hom(\Delta^n \times X, Y)$.

and given $\theta : [n] \to [m]$ get a map $Hom(X, Y)_m \to Hom(X, Y)_n$.

Prop: X a Kan complex, then if $L \subset K$, the obvious map

$Hom(K, X) \to Hom(L, X)$ is a fibration.

Pf. Omitted, see Goerss Jardine.

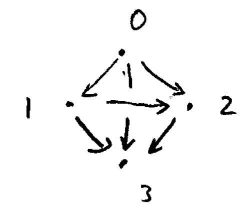
We can now reinterpret simplicial homotopy.

It is visibly just a diagram $h \overset{d_0}{\nearrow} f, \; h \underset{d_1}{\searrow} g$, $h \in \underline{\mathrm{Hom}}(X,Y)_1$, $f, g \in \underline{\mathrm{Hom}}(X,Y)_0$.

3) Simplicial Homotopy Groups.

We would like simplicial homotopy to be an equivalence relation.

But consider the two vertices 0, 1 of Δ^n.



(i.e., $K = \{*\}$, $\Delta^0 \underset{i_1}{\overset{i_0}{\rightrightarrows}} \Delta^n$.)

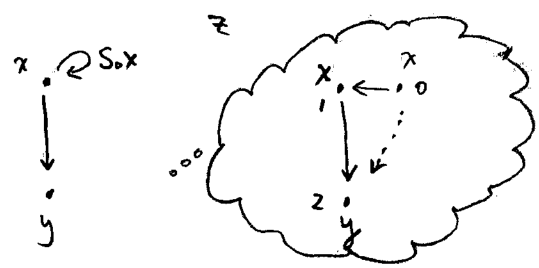
$i_0 \xrightarrow{\simeq} i_1$ but not the other way.

Prop: If X is fibrant, then simplicial homotopy of vertices $\Delta^0 \to X$ is an equiv. relation.

Pf: Reflexive: $x \in X_0$, use $s_0 x$.

Transitive: $x \to y \to z$, $x \dashrightarrow z$ because this is a map of Λ^2_1.

Symmetric:



$\square$

Cor: Since homotopies of maps $K \to X$ (rel L) are just homotopies of vertices in the fibers of $\mathrm{Hom}(K,X) \longrightarrow \mathrm{Hom}(L,X)$, we have a) homotopy of maps $K \to X$ (rel L)

b) homotopy of maps $K \to X$

are equivalence relations.