Oct 16

(note k can be anything in $\{0,\dots,n\}$)

Recall –

Def $X \in sSet$ is a Kan complex if $\forall x_0, \dots, \hat{x}_k, \dots, x_n \in X_{n-1}$ s.t. $d_i x_j = d_{j-1} x_i$ then $\exists y \in X_n$ s.t. $d_i y = x_i$, $i \neq k$. Equivalently any map $\Lambda^n_k \to X$ has a unique extension $\Delta^n \to X$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Def $X, Y \in sSet$, $f : X \to Y$ is a fibration if there's a filling

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Rmk In top spaces we usually define fibration as

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Z \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

, lifting homotopies : Z 

We're replacing this with 

So X is a Kan complex (or fibrant) if $X \to *$ is a fibration.

Also recall: $Sing(X)$ is fibrant for X a top space. (Proof by adjunction)

E.g. Let $H \in Grp^{\Delta^{op}}$ a simplicial group. Then H (or its underlying sSet) is fibrant.

Proof (Moore) Let $x_0, \dots, \hat{x}_k, \dots, x_n \in H_{n-1}$ compatible.

Strategy (to find $y \in H_n$ s.t. $d_i y = x_i$): step by step!

Start with $y = s_0 x_0$. Then $d_0 y = x_0$. By the group operation we can write down $w := s_{n-1}(x_n \cdot d_n y^{-1}) \cdot y$. Then

$$d_n w = (\underbrace{d_n s_{n-1}}_{id} x_n)(\underbrace{d_n s_{n-1}}_{id} d_n y^{-1})(d_n y) = x_n (d_n y)^{-1} (d_n y) = x_n \checkmark$$

Keep going – $\square$

Prop G a groupoid. Then the nerve BG of G is a Kan complex

Proof Observe that for a category C, $X \in sSet$, there's a bijection

$$\mathrm{Hom}(X, BC) \simeq \mathrm{Hom}(sk_2 X, sk_2 BC)$$

Why? When $X = \Delta^n$, $\mathrm{Hom}(\Delta^n, BC) \simeq BC_n \simeq \mathrm{Fun}([n], C)$

$[n] = 1 \to 2 \to \cdots \to n$ so a functor $[n] \to C$ is determined exactly by where the morphisms $i \to i+1$ go, and how they commute (triangles).

In general $X \simeq \mathrm{colim}\, \Delta^n$.

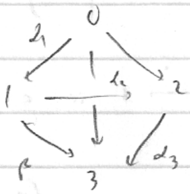
Next we want to look at extension problems

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & BG \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Case 1: $n \geq 4$. Here $sk_2 \Lambda^n_k = sk_2 \Delta^n$ (removing an $n-1$ simplex doesn't affect sk_2). Then by observation

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & BG \\ \Delta^n & \end{tikzcd} \rightsquigarrow \begin{tikzcd} sk_2 \Lambda^n_k \arrow[r] \arrow[d, equal] & BG \\ sk_2 \Delta^n \arrow[ur, "\checkmark"'] & \end{tikzcd}$$

Case 2 Λ^3_0 :

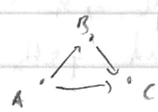


Then $d_3 d_2 d_1 = \beta d_1$, since d_1 invertible (G a grpd) then $d_3 d_2 = \beta$ so we filled in the horn

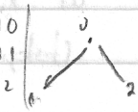
others similar $\square$

Rmk also if BC a Kan complex then C is a groupoid.

E.g. Δ^n not a Kan complex



$A \mapsto 0$
$C \mapsto 1$
$B \mapsto 2$



Don't have $C \to B$!

Philosophy Kan complexes are like spaces (Sing)
Kan complexes are like groupoids (BG)

Next, <u>simplicial homotopy</u>.

Def $f,g: K \longrightarrow X$. A homotopy of f to g is a $h: K\times\Delta^1 \to X$ such that (write $f \simeq g$)

$$\begin{tikzcd} K\times\Delta^0 \arrow[d, "id\times d^1"] \arrow[dr, "f"] & \\ K\times\Delta^1 \arrow[r, "h"] & X \\ K\times\Delta^0 \arrow[u, "id\times d^0"] \arrow[ur, "g"'] & \end{tikzcd}$$

Recall for $X,Y \in sSet$ that product defined by

$(X\times Y)_n = X_n\times Y_n$

(Weird if you think about CW complexes, it's because of <u>degenerate</u> simplices)

Note that this has a <u>direction</u> since Δ^1 has a direction

e.g. $\Delta^0 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \Delta^1$ $\qquad 0 \longrightarrow 1$

so d_0 homotopes to d_1 but not <u>vice-versa</u>

(*) HOMOTOPY NOT EQUIVALENCE RELATION.

Def $L \subseteq K \underset{g}{\overset{f}{\rightrightarrows}} X$. Then $f \simeq g$ (rel L) if $f \simeq g$ and

$$\begin{tikzcd} K\times\Delta^1 \arrow[r, "h"] & X \\ L\times\Delta^1 \arrow[u, "incl"] \arrow[r, "proj"'] & L \arrow[u, "\alpha"'] \end{tikzcd}$$

for some specified $\alpha: L \to X$. ($\alpha\circ$proj the "constant homotopy")

q: Is sSet enriched over itself? Yes. Want sSet $\ni \underline{Hom}(X,Y)$

How should we define it. Well we certainly want an adjunction:

$$Hom(X, \underline{Hom}(Y,Z)) \cong Hom(X\times Y, Z).$$

Plug in $X = \Delta^n$, get

$$\underline{Hom}(Y,Z)_n \cong Hom(\Delta^n, \underline{Hom}(Y,Z)) \cong Hom(\Delta^n\times Y, Z)$$

So this <u>defines</u> the simplices of $\underline{Hom}(Y,Z)$. And for a morphism $[m] \xrightarrow{\theta} [n]$ in Δ then have $\tilde{\theta}$, ; pull back

$$\begin{tikzcd} \Delta^m\times Y \arrow[rr] \arrow[dr] & & \Delta^n\times Y \arrow[dl] \\ & Z & \end{tikzcd}$$

Prop $X \in sSet$ fibrant. $L \subseteq K \in sSet$. Then

$$\underline{\mathrm{Hom}}(K,X) \longrightarrow \underline{\mathrm{Hom}}(L,X)$$

is a fibration

Proof Goerss–Jardine

Remember a homotopy is

$$\begin{tikzcd} K \times \Delta^0 \arrow[d] \arrow[dr, "f"] & \\ K \times \Delta^1 \arrow[r, "h"] & X \\ K \times \Delta^0 \arrow[u] \arrow[ur, "g"'] & \end{tikzcd} \iff \begin{tikzcd} & f \\ h \arrow[ur, "d_0"] \arrow[dr, "d_1"'] & \\ & g \end{tikzcd} \text{ in } \underline{\mathrm{Hom}}(K,X)$$

Comparison of sSet	and Spaces
Adjunction $\underline{\mathrm{Hom}}(X, \underline{\mathrm{Hom}}(Y,Z)) \cong \underline{\mathrm{Hom}}(X \times Y, Z)$ for free.	Adjunction not free! needed to define $\underline{\mathrm{Hom}}$ and restrict spaces to compactly generated
Homotopy, equivalence relation not free. Must restrict to Kan complexes.	Homotopy is an equivalence relation for free
Prop at top of page $K \subset L$ a cofibration	$K \hookrightarrow L$ a cofibration (K,L) has HEP. Then $\mathrm{Map}(K,X) \longrightarrow \mathrm{Map}(L,X)$ is a fibration

Both have notions of good inclusions (cofibrations) and good projections (fibrations)

$X \longrightarrow *$
e.g. $*/X = \Sigma X$

q: Why do we need fibrations and cofibrations?
a: gives good notion of limits and colimits in homotopy theory

Eg. $f: A \longrightarrow X$ any map of spaces. What should X/A be?
In homotopy land, replace f by a cofibration: factor f as $A \xrightarrow{\text{cofibration}} M_f \xrightarrow[\sim]{\text{homotopy equiv.}} X$. $X/A := M_f/A$