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Table

Main object of study: homotopy category of finite CW spectra

$$[X, Y] = \text{colim}_n [\Sigma^n X, \Sigma^n Y]_{\text{stable}}$$

Main tool: • complex oriented (or orientable) cohomology theories
• algebraic geometry of formal groups.

- § 1. complex oriented theories
- § 2. MU
- § 3. Hough, $k(n)$, LEFT
- § 4. Nilpotence result
- § 5. Chromatic calculus

§ 1. Cplx oriented: If $V \xrightarrow{f} X$, there is a natural Thom class

$$\begin{array}{ccccc} E^{2n}(F, F) & \leftarrow & E^{2n}(V, V_0) & \rightarrow & E^{2n}(V) \cong E^{2n}X \\ \parallel & & \downarrow & & \downarrow \eta^* \\ E^{2n}(\mathbb{C}, \mathbb{C}) & \cong & E^0 & & U \longrightarrow e(V) \text{ Euler class, } e(V) \in E^{2n}X. \end{array}$$

Cysin square gives $E^*(\mathbb{CP}^\infty) \cong E^*[\langle e \rangle] \cong \lim E^*(\mathbb{CP}^n)$
" $B\mathbb{Z}/2$

If ξ_1, ξ_2 are line bundles $e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2))$ where

F.G.L. of E $x +_E y = F(x, y) \in E^*(X, Y) \cong E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$.

Another choice of orientation: $e'(\xi) = \varphi(e(\xi))$ where $\varphi \in E^*(\mathbb{Z})$ has $\varphi(1) \neq 0, \varphi(1) = 1$.
(stable)

$\varphi(x +_E y) = \varphi(x) +_E \varphi(y)$ I so of F.G.L.s.

Example: $H^*(-, \mathbb{Q})$ $x +_F y = x + y$

$(1+x)(1+y) - 1$

K^+

$x +_F y = \beta^{-1}(\beta x + \beta y) = \beta^{-1}((1+\beta x)(1+\beta y) - 1)$

completed at p $\text{Aut}(\hat{G}_m) \cong \mathbb{Z}_p^\times$ where $\varphi_a(x) = (1+x)^a - 1$ "up to β "
(L-2)

Theorem: There exists L and FGL/L so that $H_{\text{an}}(L, R) \cong FGL_R/R$
 $f \mapsto f^*F$

$L \cong \mathbb{Z}(x_1, x_2, \dots)$ (non-canonically)

MU^* is complex orientable and any choice of orientation gives an iso $L \xrightarrow{\sim} MU_*$.

2

§2. Haight, LEFT, Mantra k-theory Fix a prime p. and FGL F

Let $[p](x) = x +_F \dots +_F x$. Then $[p]_k(x) = (1+x)^P - 1$.

Modulo p $[p](x) \equiv \sum_{i=1}^P f_i(x^P) = v_1 x^P + \dots$
 $\equiv \sum_{i=1}^P f_i(x^{P^2}) = v_2 x^{P^2} + \dots$
 v_1

etc. This defines a sequence of elements $p, v_1, v_2, \dots$

Definition: R a $\mathbb{Z}_p$ algebra. F is Landweber exact if $p, v_1, v_2, \dots$ form a regular sequence

F has ht n if $p = v_1 = \dots = v_{n-1} = 0$ and v_n is a unit.

L.E.F.T. (1) If F is a L.E., then is a homotopy (x oriented) homology theory $E(R, F)$ and

$E(R, F)_* = R_*$ and F.G.L. F .

(2) If F is p -typical (not defined here) $[p]_F = px +_F v_1 x^P +_F \dots +_F v_n x^{P^n}$ and F is determined by the v_i .

Ex: $R = \mathbb{Z}_p[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$ with $[p] = px +_F \dots +_F v_n x^{P^n}$. Land Exact

Mantra k-theory $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}] \cong \mathbb{F}_p[v_n^{\pm 1}]$ $\xleftarrow{\substack{v_n = \mu^{P^{n-1}} \\ \text{field element}}} \mathbb{F}_p[v_n^{\pm 1}]$ $[p](x) = v_n x^{P^n}$

Ex: $R = \underbrace{W(\mathbb{F}_p)}_0[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$. These are the Mantra Tot or Mantra E-theory

Theorem: (Hopkins-Smith). Let X be a finite CW spectrum, $K(n)_* X = 0$. Then central

X is a self map $f: \Sigma^s X \rightarrow X$ with $K(n)_* f = v_n^t$ & $s = \text{ht}(p^n - 1)$.

and the

$\mathbb{Z}_m[v_n] \leftarrow \mathbb{Z}_m[v_n^{\pm 1}] \rightarrow \mathbb{Z}[X, X]_*$

are F isomorphisms.

3

§3. Chromatic convergence: Let E be a homology theory. Z is E -local if

$$LA, Z] = 0 \quad \text{when } E_* Z = 0$$

$Z \xrightarrow{\eta} L_E Z$ is E -localization of $E_* Y$ is an iso and $L_E Z$ is local. (Exists ξ map Bouffard '79, '75)

Then

$$L_n = L_{E(n)} = L_{K(n) \vee \dots \vee K(n)}. \quad \text{Then there are maps } L_n X \rightarrow L_{n-1} X.$$

and if X is finite CW

$$X \rightarrow \text{colim } L_n X. \quad (\text{H-R. chromatic convergence})$$

is $H_*(-, \mathbb{Z}_{(p)})$ localization. Furthermore there is a pull-back square

$$\begin{array}{ccc} H_* X & \xrightarrow{\text{H.S.}} & H_* L_{K(n)} X \\ \downarrow & & \downarrow \\ L_n X & \rightarrow & L_{K(n)} X (= L_n L_{K(n)} X) \\ \downarrow & & \downarrow \\ L_{n-1} X & \rightarrow & L_{n-1} L_{K(n)} X \end{array}$$

Chromatic Spectral convergence

§4. Computability: $L = MU_*$ represents $R \rightarrow FG.L$ over R .

Then $(f, \phi) \in MU_*$ represents $R \rightarrow (f, \phi)$

Then (LEFT rule) $E_1 \times E_2$ represents $E(R_1, F_1), E(R_2, F_2)$ and one L.E.

$$(E_1) \times E_2 \text{ repn } (f: R_1 \rightarrow S, g: R_2 \rightarrow S, \phi: f^* F_2 \rightarrow g^* F_1).$$

Example: $E_* MU$ represents $(f: R \rightarrow S, \phi: f^* F \rightarrow \text{anything}) = E_* [a_1, a_2, \dots]$.

$$(MU_*, MU_* MU)(S) = \left\{ \frac{FGLS/R}{\text{Isos of FGLs}} \right\}. \quad \text{Happy algebraic, "computable"}$$

Now there are two maps $MU_* \xrightarrow[\eta_R]{\eta_L} MU_* MU$. (source and target).

If X is a spectrum, $MU_* X \cong MU_* S_* X \rightarrow MU_* MU_* X$

$$MU_* X \rightarrow MU_* M \circ MU_* X \leftarrow \text{canonical structure.}$$

$\downarrow MU_*$
 q.r.

$(= \text{q.r. sheaf on the pro-scheme})$

attempts to do ANSS:

$$\text{Ext}_{M \otimes M}^s(\Sigma^t M \otimes X, M \otimes Y) \Rightarrow [\Sigma^{t+s} X, L_{M \otimes M} Y] = \pi_{t+s} Y.$$
$$G_n = \text{Aut}(\mathbb{F}_{p^n}, \Gamma_n)$$
$$H^s(\mathbb{G}_n, (E_n)_+, X) \Rightarrow \prod_{t \in S} L_{K(t)} X.$$
$$\text{Ext}_{H_{\#}H}^1(\Sigma_{\#}^* H_{\#}, H_{\#}, \varphi) \xrightarrow{\cong} H^1(\overline{G}^{\times}, (E_{\#})_{\#}) =$$

$$\pi_{*-1} S^0(p) \rightarrow \pi_{*-1} L_{K(1)} S^0 \quad (* \geq 0).$$

detects the way of the J-haemoglobin. || "Chromatic Spectral Sequence".