Lecture 1:An overview of chromatic homotopy theory

Paul Goerss

Transcribed by Paul VanKoughnett

1 Complex oriented cohomology theories and formal group laws

Let E be a commutative ring spectrum and $q : V \to X$ a $\mathbb{C}^n$-bundle. We let V_0 be the complement of the zero section of V, $F \cong \mathbb{C}^n$ a fiber, F_0 the complement of 0 in F. A **complex orientation** or **Thom class** for E is an element $v \in E^{2n}(V, V_0)$ such that under the composition $E^{2n}(V, V_0) \to E^{2n}(F, F_0) \cong E^{2n}(\mathbb{C}^n, \mathbb{C}^n - \{0\}) \cong E^0$, v gets sent to a unit of the ring $\pi_0 E$. The **Euler class** of v is a class $e(v) \in \widetilde{E}^{2n}(X)$ such that $q^*(e(v))$ is the image of v in $E^{2n}(V)$.

Using the Gysin sequence, it can be shown that $E^*\mathbb{C}P^\infty \cong E^*[[e(\zeta)]]$, where ζ is the topological line bundle. (Here and afterwards, we write E^* for the ring $\pi_* E$.)

This allows us to construct a formal group law from any E. Namely, if ξ_1 and ξ_2 are two line bundles over a space X, then $e(\xi_1 \otimes \xi_2) = F_E(e(\xi_1), e(\xi_2))$, where F_E is a power series over E^* in two variables. We also write this as $e(\xi_1) +_{F_E} e(\xi_2)$. Now, by the standard properties of the tensor product such as associativity and commutativity up to isomorphism, the operation described is a smooth 1-parameter formal Lie group with a coordinate – in other words, a **formal group law**, or **FGL**.

If we change the choice of Thom class, we also change the Euler classes and thus the FGL. If e and e' are two choices of Euler class maps and F and F' the associated FGLs, then e and e' will satisfy a relation $e'(\xi) = \phi(e(\xi))$, where $\phi(x) \in E^*[[x]]$ satisfying the following two conditions:

- $\phi(0) = 0$, since ϕ must be in the augmentation ideal of $E^*[[x]]$, and
- $\phi'(0)$ is a unit, since the Thom class must still be sent to a unit in E^0.

Thus, ϕ is of the form

$$\phi(x) = a_0 x + a_1 x^2 + a_2 x^3 \cdots$$

with a_0 a unit (note the odd indexing), and we have

$$\phi(x +_F y) = \phi(x) +_F \phi(y).$$

Such a ϕ is called an **isomorphism** of FGLs. (If $a_0 = 1$, it is a **strict isomorphism**.)

There is a category (indeed, a groupoid) of FGLs and isomorphisms over a fixed ring. At this point, there are basically three examples available to us.

1. $H\mathbb{Q}^*$, ordinary cohomology. The Chern class of a tensor product of line bundles is just the sum of their Chern classes, so $x +_F y$ is just $x + y$.
2. K^*, complex K-theory. The Euler class is given by $e(\xi) = \beta(\xi - 1)$ where $\beta \in K^2$ is the Bott element. Thus $x +_F y = x +_F y + \beta xy$.

2bis Real K-theory KO^* is not complex orientable.

3. MU^*, complex cobordism. Surprisingly, the coefficient ring both has a simple structure and represents the FGLs functor, in the sense given by the following theorem.

Theorem 1.1 (Lazard-Quillen)**.** *There is a commutative ring L, the **Lazard ring**, and an FGL F_u over L, such that there is a natural isomorphism* $\mathrm{Hom}(L, R) \cong \{$*FGLs over* $R\}$*, given by sending a map $f : L \to R$ to $f_* F_u$. Moreover, there is a non-canonical isomorphism $L \cong \mathbb{Z}[x_1, x_2, \dots]$. Finally, $MU^* \cong L$, again non-canonically, but with any choice of orientation giving an isomorphism.*

2 Height and Landweber exactness

Fix a prime p, a ring R, and an FGL F over R. We define

$$[p](x) = \underbrace{x +_F \cdots +_F x}_{p \text{ times}} = px + \cdots .$$

Mod p, this is a power series in x^p, of the form $v_1 x^p + \cdots$ for some $v_1 \in R$. Mod (v_1, p), we get a power series in x^{p^2}, of the form $v_2 x^{p^2} + \cdots$, and so on. We thus define elements $v_n \in R$ for each $n \in \mathbb{N}$.

Definition 2.1. F has **height** $n \geq 1$ if $p = v_1 = \cdots = v_{n-1} = 0$ and v_n is a unit. (In particular, R must have characteristic p.)

F is **Landweber exact** if $p, v_1, v_2, \cdots$ is a regular sequence. This means that each element in the sequence is a nonzerodivisor mod the previous elements.

Theorem 2.2 (Lazard)**.** *If R is an algebraically closed field of characteristic p, then two FGLs are isomorphic iff they have the same height.*

Theorem 2.3 (Landweber exact functor theorem)**.** *If F is a Landweber exact FGL over a ring R, then the functor*

$$X \mapsto R \otimes_{MU_*} MU_* X,$$

is a homology theory, written $E(R,F)_* X$, and $E(R,F)_* \cong R_*$. (Here the tensor product is computed using the map $MU_* \to R$ determined by F.)

Example 2.4. Let $R = \mathbb{Z}_{(p)}[v_1, \ldots, v_n^{\pm 1}]$, and choose an FGL F over R such that

$$[p]_F(x) = px +_F v_1 x^p +_F \cdots +_F v_n x^{p^n}$$

for some n. (That such an F exists is guaranteed by the theory of p-typical FGLs, on which presumably more later.) Then $E(R,F)_*$ is **Johnson-Wilson homology**, written $E(n)_*$.

Example 2.5. Morava K-theory is a sequence of homology theories with $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ and $[p]_F(x) = v_n x^{p^n}$. These are 'field theories' – every graded module is free, the Künneth map is an isomorphism, and so on.

Example 2.6. Morava E-theory is a sequence of homology theories with $(E_n)_* = w(\mathbb{F}_{p^n})[[\mu_1, \ldots, \mu_{n-1}]][\mu^{\pm 1}]$, with $\mu_1, \ldots, \mu_{n-1}$ in degree zero. Here w is the Witt vector functor. For example, $w(\mathbb{F}_p) = \mathbb{Z}_p$.

Theorem 2.7 (Hopkins-Smith)**.** *Let X be a finite CW-spectrum and $K(n-1)_* X = 0$. Then for some t, for $s = 2t(p^n - 1)$, there exists a nonzero map $f : \Sigma^s X \to X$ such that*

$$K(n)_* f = v_n^t : K(n)_* \Sigma^s X \to K(n)_* X,$$

and $f \in \operatorname{End}_*(X)$ is central and unique up to nilpotence. Moreover, the center $Z(\operatorname{End}_* X) \cong \mathbb{Z}/(p^k)[v_n]$ up to nilpotence for some k, in that there is a map between the two rings whose kernel and cokernel are nilpotent.

Example 2.8 (Adams)**.** Let $p > 2$. We let $V(0)$ be the Moore spectrum $S^0 \cup_p e^1$. Here $t = 1$, meaning that there is a nonzero map

$$v_1 : \Sigma^{2(p-1)} V(0) \to V(0).$$

In particular, for any t we get a nonzero map $\alpha_t : S^{2t(p-1)} \to S^1$ given by including $S^{2t(p-1)}$ as the bottom cell of $\Sigma^{2(p-1)} V(0)$, applying v_1^t, and collapsing the bottom cell of $V(0)$. For each t, α_t is an element of order p in $\operatorname{im} J$. This was the first example of an infinite family of elements of $\pi_* S$! For more see Adams, *$J(x)$ IV*, Topology 1966.

3 Chromatic convergence

Let E be a homology theory. A spectrum Z is **E-local** if $[A, Z] = 0$ whenever $E_*A = 0$. A map $X \to L_E X$ is an **E-localization** if it is an E_*-isomorphism and $L_E X$ is E-local.

Theorem 3.1 (Bousfield). *A functor sending each spectrum X to an E_*-localization of X always exists.*

Let $L_n X = L_{E(n)} X = L_{K(0)\vee\cdots\vee K(n)} X$ (the second equality is a theorem). There are maps $L_n X \to L_{n-1} X$, and if X is a finite CW-spectrum, $X \to \operatorname{holim} L_n X$ is an $H\mathbb{Z}_{(p)}$-localization. This is **Hopkins-Ravenel chromatic convergence**. Also,

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

is a homotopy pullback square. The fibers along the vertical maps are $M_n X \xrightarrow{\sim} M_n L_{K(n)} X$, the **monochromatic piece**.

4 The Adams-Novikov Spectral Sequence

Let $G_1 = E(R_1, F_1)$ and $G_2 = E(R_2, F_2)$ for F_1, F_2 Landweber exact. By the LEFT, $(G_1)_* G_2$ represents

$$S \mapsto \{(f_1 : R_1 \to S, f_2 : R_2 \to S, \phi : f_* F_1 \xrightarrow{\cong} g_* F_2)\}.$$

Letting $G_2 = MU$, $(G_1)_* MU$ represents

$$S \mapsto \{(f : R_1 \to S, \Xi \text{ an FGL over } S, \phi : f_* F_1 \to \Xi.$$

Letting both be MU, $(MU)_* MU$ represents the set of isomorphisms of FGLs!

Thus the pair $(MU_*, MU_* MU)$ represents the functor sending a ring S to the groupoid of FGLs over S and their isomorphisms. This is called an **affine groupoid scheme**, and its representing object a **Hopf algebroid**, by way of analogy with Hopf algebras representing affine group schemes.

In general, $(E_*, E_* E)$ is a Hopf algebroid for any Landweber exact theory E, though we will focus on $E = MU$.

The axioms for a Hopf algebroid are induced by the axioms for a groupoid – indeed, it helps to just think of MU_* and $MU_* MU$ as the objects and morphisms of a groupoid. For example, there are 'source and target' maps $\eta_L, \eta_R : MU_* \to MU_* MU$. Likewise, there are 'composition,' 'identity' and 'inverse' maps, and these must satisfy a bunch of diagrams. For details, see Appendix 2 of Ravenel's green book.

If X is any spectrum, there is a map

$$\begin{tikzcd} \pi_*(MU \wedge X) \arrow[r, "\cong"] \arrow[d, equal] & \pi_*(MU \wedge S^0 \wedge X) \arrow[r] & \pi_*(MU \wedge MU \wedge X) \arrow[d, "\cong"] \\ MU_* X \arrow[rr, "\psi_X"'] & & MU_* MU \otimes_{MU_*} MU_* X. \end{tikzcd}$$

This makes $MU_* X$ a **comodule** over the Hopf algebroid $(MU_*, MU_* MU)$. Another way to say this is hat $MU_* X$ is a quasi-coherent sheaf on the affine groupoid scheme of FGLs.

The map $[X, Y] \to \operatorname{Hom}_{MU_* MU}(MU_* X, MU_* Y)$ (the right-hand side is maps of comodules) extends to the **Adams-Novikov spectral sequence**

$$E_2^{s,t} = \operatorname{Ext}^s_{MU_* MU}(\Sigma^t MU_* X, MU_* Y) \Rightarrow [\Sigma^{t-s} X, L_{MU} Y].$$

This is a second-quadrant spectral sequence, so convergence is an issue. However, localization is not: it is a theorem that $L_E Y$ only depends on E_0, so that $L_{MU} Y = L_{H\mathbb{Z}} Y$, which is just Y if Y is connective. In particular, if Y is connective and $X = S$, the SS converges to $\pi_{t-s}(Y)$.[1]

[1] In the following two examples, the note-taker lost track of what was going on, and apologizes in advance for the resulting opacity.

Example 4.1. $(E_n)_*E_n = \mathrm{Hom}(\mathbb{G}_n, (E_n)_*)$, where $\mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_{p^n}, \Gamma_n)$ and Γ_n is the height n formal group. We have a spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_n, (E_n)_t X) \Rightarrow \pi_{t-s} L_{K(n)} X.$$

Example 4.2. Let $p > 2$. For $* > 0$, there is a square

$$\begin{tikzcd} \mathrm{Ext}^1_{MU_*MU}(\Sigma^* MU_*, MU_*)_{(p)} \arrow[r, "\cong"] & H^1(\mathbb{Z}_p^\times, (E_*)_* \arrow[r, equal] & \mathbb{G}_1 \\ (\pi_{*-1} S^0)_{(p)} \arrow[u] \arrow[r] & \pi_{*-1} L_{K(1)} S^0 \arrow[u, "\cong"'] & \end{tikzcd}$$

which detects the image of J.