

Paul v K.

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Ring spectra + Bousfield localization:

spectra:

Def: a sequence of spaces $\{X_n\}$ with maps $\Sigma X_n \rightarrow X_{n+1}$ Ex: X space; suspension spectrum $X_n = \Sigma^n X$; maps ident.

Maps of spectra: $\Sigma X_n \rightarrow X_{n+1}$ except that maps only
 $\downarrow \quad \downarrow$ defined cofinally
 $\Sigma Y_n \rightarrow Y_{n+1}$

Complaint: There is no nice category of spectra.Thm (Lewis) There is no cat. of spectra satisfying

1. $\wedge$ symm. monoidal
2. Σ^∞ : spaces $\rightleftarrows$ spectra: Ω^∞ adjunction
3. natural iso $\Omega^\infty \Sigma^\infty X \xrightarrow{\cong} \operatorname{colim}_n \Omega^n \Sigma^n X$
4. $\Sigma^\infty S^0$ is unit of $\wedge$
5. at least one of the functors $\Omega^\infty, \Sigma^\infty$ lax monoidal
 (e.g. $\Sigma^\infty (X \wedge Y) \rightarrow \Sigma^\infty X \wedge \Sigma^\infty Y$)

Comment: Will focus not on one of the modern categories of spectra (like S-modules), but just use Adams' construction + work in homotopy category

CW-spectra: Each X_n CW-cx; each $\Sigma X_n \rightarrow X_{n+1}$ is cellular.

- finite if each X_n is a finite CW-cx
- subspectrum of a CW-spectrum X is a subcx $Y_n \subseteq X_n$ s.t.

$$\begin{array}{ccc} \Sigma Y_n & \longrightarrow & Y_{n+1} \\ \cap & & \cap \\ \Sigma X_n & \longrightarrow & X_{n+1} \end{array} \quad \text{commutes.}$$

(2)

A subspectrum is cofinal if every cell of X_n eventually lands in a cell of Y .

maps $X \rightarrow Y$ are defined on a cofinal subspectrum of X : i.e. $\exists$ a subspectrum $W \subset X$ cofinal; maps $W_n \xrightarrow{f_n} Y_n$
 s.t. $\Sigma W_n \rightarrow \Sigma Y_n$ commutes

$$\begin{array}{ccc} \downarrow & & \downarrow \\ W_{n+1} & \rightarrow & Y_{n+1} \end{array}$$

and two maps from W, W' cofinal subspectra of X are equivalent if they agree strictly on a cofinal subspectrum of X .

(Note: this ~~definition~~ definition motivated by Kan-Priddy map:

$\mathbb{R}P_+^{n-1} \xrightarrow{\text{reflect.}} O(n) \rightarrow \Omega^n S^n$ adjoint to $\Sigma^n \mathbb{R}P_+^{n-1} \rightarrow S^n$
 which assembles to $\Sigma^\infty \mathbb{R}P_+^\infty \xrightarrow{\tau} S^0$

which is onto in htpy, but you can never extend to all of $\mathbb{R}P^\infty$ by Sullivan conjecture)

Stable homotopy category:

homotopies: Given spectrum X , create new spectrum

$X \wedge I_+$ with $(X \wedge I_+)_n = X_n \wedge I_+$

$\Sigma(X_n \wedge I_+) = \Sigma X_n \wedge I_+ \rightarrow X_{n+1} \wedge I_+$

Given fig: $X \rightarrow Y$, a homotopy $f \sim g$ is a map

$$\begin{array}{ccc} \cup & & \\ W & \xrightarrow{H} & Y \end{array}$$

 W cofinal subspectrum, $H: X \wedge I_+ \rightarrow Y$

restricting to $f + g$ as usual.

Defn: Stable homotopy cat. is cat. where objects are spectra & maps are homotopy classes of maps of spectra.

Defn: The sphere spectrum is $(S)_n = S_n$ with $\Sigma S_n \rightarrow S_{n+1}$ ident.

Notation: $[X, Y]$ = maps is stable htpy cat.

$[X, Y]$ = graded set with $[X, Y]_n = [\Sigma^n X, Y]$

Hence $[S, X]_* = \pi_* X$.

if $X = \Sigma^\infty A$, then $[S, X]_* = \text{colim}_{k \rightarrow \infty} \pi_{k+n} \Sigma^k A$ (usual stable htpy grps of A)

Things to do with spectra:

1. suspend them: $(\Sigma X)_n = X_{n+1}$

2. loop them: $(\Omega X)_n = X_{n-1}$

These satisfy usual adjunction, and are inverse.

Thus Eckman-Hilton argument shows these are abelian groups.

In fact stable htpy cat is triangulated:

3. cofiber seq: $X \xrightarrow{f} Y$ yields

$$X \xrightarrow{f} Y \rightarrow Cf = Y \cup_f CX$$

and these are also fiber sequences:

$$\dots \rightarrow \Omega Cf \rightarrow X \rightarrow Y \rightarrow Cf \rightarrow \Sigma X \rightarrow \Sigma Y \rightarrow \dots$$

Mapping into a spectrum Z yields a long-exact seq:

$$\leftarrow [X, Z] \leftarrow [Y, Z] \leftarrow [Cf, Z] \leftarrow$$

or map Z into it:

$$\rightarrow [Z, X] \rightarrow [Z, Y] \rightarrow [Z, Cf] \rightarrow \dots$$

These are "about as useful as you'd expect"

4. smash products: $X \wedge Y$ Adams: $\Sigma^2(X_{n+1} \wedge Y_n) \rightarrow X_{n+1} \wedge Y_{n+1}$
works poorly.

very painful from this perspective

know: $\wedge$ exists, comm. & assoc. on htpy cat

S is unit; & have $(X \wedge A)_n = X_n \wedge A$; $X \wedge A \cong X \wedge \Sigma^\infty A$.

(but we won't define it.)

Ring spectra:

• Monoid object in stable htpy cat:

$$E \wedge E \xrightarrow{\text{product}} E \quad S \xrightarrow{\text{unit}} E$$

~~Assoc.~~ (assoc. upto htpy)

+ diagrams in stable htpy cat.

• A module spectrum over E is F with $E \wedge F \rightarrow F$

(again, vital upto htpy...)

(co)homology E a spectrum, X space or spectrum.Defn: The E homology groups of X are $E_*X = [S, E \wedge X]_*$ The E cohomology groups of X are $E^*X = [X, E]_{-*}$ Ex: If A is an abelian group, define HA with $(HA)_n = K(A, n)$,
 $\Sigma HA_n \rightarrow HA_{n+1}$ are adjoint to $K(A, n) \cong \Omega K(A, n)$.This is an Eilenberg-MacLane spectrum for A .

$$\text{Map}(X, K(A, n)) = H^n(X; A)$$

$$\text{so } HA^*X = [X, HA]_* = H^*(X; A).$$

Ex: $E=S$. $E_*X = [S, X]_* = \pi_*X$.

$$S^*X = [X, S]_* = \text{cohomotopy groups of } X.$$

Comment: These satisfy Eilenberg-Steenrod axioms and thus form a generalized (co)-homology theory.Thm (Brown representability; Whitehead)

- Every homology theory is of the form $E_*(-)$ for some E ; likewise cohomology.
- Ring spectra ~~produce~~^{yield} good cup products.

In general, have external products

$$E^*X \otimes F^*X \rightarrow (E \wedge F)^*X \quad \text{so if } E=F=\text{ring spectrum}$$

$$\text{get } E^*X \otimes E^*X \rightarrow (E \wedge E)^*X \rightarrow E^*X. \quad \text{product.}$$

Cor: $[X, Y]_*$ is a homology theory in Y , and isthus representable: $[X, Y]_* \cong [S, X^\vee \wedge Y]$ where $X^\vee$ is the Spanier-Whitehead dual of X $(X^\vee \text{ gives closed symm. monoidal structure on stable htpy cat.})$

Ex: Let A be an abelian grp, with free resolution

$$0 \rightarrow R \rightarrow F \rightarrow A \rightarrow 0$$

then the Mooré spectrum SA is the cofiber

$$VS \rightarrow VS \rightarrow SA$$

Also define $E_* SA = E_* A$ (Note: $HZ_n A = H_n A$).

By long exact seq: $\pi_* SA = 0$ if $* < 0$

$$\pi_0 SA = H_0 SA = A$$

$$\text{and } H_n SA = 0 \quad n \geq 1.$$

Thrm: If E is a spectrum, A an abelian group, there are natural short exact sequences.

$$0 \rightarrow E_n(X) \otimes A \rightarrow E_n(X) \rightarrow \text{Tor}(E_{n-1}(X), A) \rightarrow 0$$

$$0 \rightarrow E^n(X) \otimes A \rightarrow (EA)^n X \rightarrow \text{Tor}(E^{n+1}(X), A) \rightarrow 0.$$

Note these don't split.

Bousfield localization:

Adams SS: $\text{Ext}_{E_* E}^*(E_* X, E_* Y) \Rightarrow "[X, Y]"$

but E_2 -term only sees things visible to E , so it can only converge to things E detects.

So we consider E_* -localization, which makes maps that E sees as equivalent equivalent.

(ie. maps that are isos under E_*)

Def: X is E_* -acyclic if $E_* X = 0$

X is E_* -local if when $A \rightarrow B$ is an E_* -equivalence, $[B, X] \rightarrow [A, X]$ is an iso.

Equivalently, $[C, X] = 0$ when C is E_* -acyclic.

A localization of X is an E_* -equivalence $X \rightarrow L_E X$

where $L_E X$ is E_* -local.

(7)

under wedges, summands, cofibers, suspensions, desuspensions...

WTS: every E_* -acyclic spectrum is in $\mathcal{C}(A)$.

Let K be as before. Let $\{B_\alpha\}$ be a choice of representatives of equivalence classes of E_* -acyclic spectra with $\leq K$ cells.

Let $A = \bigvee B_\alpha$. Note A is E_* -acyclic; each $B_\alpha \in \mathcal{C}(A)$.

If X is E_* -acyclic, construct

$A = C_0 \subseteq C_1 \subseteq \dots \subseteq C_\gamma = X$ via cofibers seq. $C_\gamma \rightarrow C_{\gamma+1} \rightarrow B_\alpha$
 note: we construct limit as a cofib seq. so we're still in $\mathcal{C}(A)$. $C_\alpha = \lim_{\beta < \alpha} C_\beta$ of λ limit ordinal.

Now we need only construct A_* trivializations to get L_E :

want $X \rightarrow L_E X$ s.t. $[A, L_E X]_* = 0$.

→ can cone off all maps from A ; maybe transfinitely

→ use small object argument to get functoriality.

Ex: $E = H\mathbb{Z}_{(p)}$. $\pi_*(L_{H\mathbb{Z}_{(p)}} X) = \pi_* X \otimes \mathbb{Z}_{(p)}$.

Thm $L_{H\mathbb{Z}_{(p)}} X = X \wedge \mathbb{Z}_{(p)}$ (Bousfield) for connective X .

(indeed localization of conn. spectrum only depends on π_0).

can reduce to E with $\pi_0 E = \bigoplus \mathbb{Z}(p)$ or $\mathbb{Z}_{(p)}$.

Note: Adams SS actually converges to $[X, L_E Y]$.

Also we have pullback

$$\begin{array}{ccc} L_{S\mathbb{Z}} X & \longrightarrow & L_{S\mathbb{Q}} X \\ \downarrow & & \downarrow \\ \prod_p L_{S\mathbb{Z}/p} X & \longrightarrow & L_{S\mathbb{Q}} \left(\prod_p L_{S\mathbb{Z}/p} X \right) \end{array} \quad \text{nb. } L_{S\mathbb{Z}} X = X \text{ if } X \text{ conn.}$$

(all localizations trivial if $X = K$ though)