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MU - complex cobordism, etc.

Defn: ~~Let~~ Let M_1, M_2 be almost complex n -mflds. (have almost α = stable tangent bundle ~~admits~~ ^{has} a complex structure.) for

Defn: Let X be a space, and let M_1 and M_2 be almost complex manifolds, as above, with $f_1: M_1 \rightarrow X$ and $f_2: M_2 \rightarrow X$. f_1 and f_2 are bordant if there exists an $(n+1)$ mfld W^{n+1} whose boundary $\partial W = M_1 \amalg M_2$ a map $H: W \rightarrow X$ restricting to $f_1 + f_2$ on boundary. (compatible w/ almost α structure.)

Defn: $MU_n(X) = \{f: M^n \rightarrow X\} / \text{bordism}$

Pontryagin (a Milnor).

Thm (Thom - ~~Milnor~~): $MU_*(pt) \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^1, \mathbb{CP}^2, \dots]$

In fact, $MU_*(pt)$ is torsion free and thus

Milnor $MU_*(pt) = \mathbb{Z}[t_1, t_2, \dots \mid |t_i| = 2i]$.

Note: can chose t_n s.t. $(n+1)t_n = \mathbb{CP}^n$. (Misheuko)

Note: use mult = ^{cart} product + add = connect sum for ring structure (could use add = disjoint union)

Note: The Lazard ring $L \cong \mathbb{Z}[x_i \mid |x_i| = 2i]$.

There's no canonical isomorphism $L \cong MU_*$, but a choice of orientation on MU induces such an isom.

Defn: A multiplicative cohomology theory E^* is complex orientable if $i: \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ induces $i^*: E^*(\mathbb{CP}^\infty) \rightarrow E^*(\mathbb{CP}^1)$

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as s.t. i^* is surjective. Equivalent to saying
for every cx n -plane bundle, $\exists u \in E^{2n}(V, V_0)$ (a Thom class)
Easy to see MU_* is cx orientable

(Quillen?)
Thrm: For every choice of orientation, the FGL on MU_* induces an isom. $L \cong MU_*$.

Defn of MU^* : $MU^*(X) = \{ Z \xrightarrow{\quad} X \times \mathbb{R}^N \text{ almost } cx \text{ orientable } \}$
 $\downarrow \quad \quad \quad \downarrow \pi_*$
 $X \quad \quad \quad \text{normal bundle}$
 $Z_1 \times mfd.$ i.e. almost cx structure on normal bundle
 $Z \xrightarrow{\quad} B\mathbb{O}$
 where $Z_1 \rightarrow W$
 $Z_1 \sim Z_2$ if $\downarrow \quad \downarrow$ is cobordism
 $X \times I \rightarrow X \times \mathbb{R}$

Defn: For F a FGL/ $\mathbb{R}$, the logarithm is $\log_F(x) \in \mathbb{R} \otimes \mathbb{Q}[[x]]$
 $\log_F(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial t}(t, 0)}$ Note: $\log_F(x+y) = \log_F(x) + \log_F(y)$
 Over $\mathbb{Q}$ alg, a FGL determined by $\log$

Prop: For some choice of orientation, $\log_F(x) \in MU_*$
 $\log_F(x) = \sum_{n \geq 0} \frac{[aP^n]}{n+1} x^{n+1}$

Lemma:
if Quillen's Thrm holds for one orient, it holds for all orientations

Pf of Quillen's Thrm: Let F_u be the universal FGL/ $\mathbb{Q}$ and let $q: L \rightarrow MU_*$ defined by the FGL of MU_* .
 proposition under consideration of the prop.

Step 1: $q \otimes \mathbb{Q}$ is an iso.

$F_u \otimes \mathbb{Q}$ has $\log \sum_{n \geq 0} \frac{P_n}{n+1} x^{n+1}$ $F \otimes \mathbb{Q}$ has $\log \sum_{n \geq 0} \frac{[aP^n]}{n+1} x^{n+1}$

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Then $\varphi(p_n) = [\mathbb{CP}^n]$. (a by comparison of ranks, the p_n must generate L .)

Hence φ is injective. (b/c L torsion free).

Step 2: φ is ~~isomorphism~~ surjective

Consider the Milnor hypersurfaces $H_{ij} \subset \mathbb{CP}^i \times \mathbb{CP}^j$ with $i \leq j$, cut out by $x_0 y_0 + \dots + x_i y_i = 0$.

Claim these generate MU_* .

let $f_i: H_{ij} \rightarrow \mathbb{CP}^i$, $f_j: H_{ij} \rightarrow \mathbb{CP}^j$, $\xi_i = f_i^* \gamma_i$, $\xi_j = f_j^* \gamma_j$ where γ_i, γ_j are the tautological bundles.

let $f: H_{ij} \rightarrow \mathbb{CP}^\infty$ be the map classifying $\xi_i \oplus \xi_j$.

let $\pi: H_{ij} \rightarrow pt$, ~~$\pi_*(\xi_i \otimes \xi_j)$~~ ~~$\pi_*(\xi_i \otimes \xi_j)$~~

Lemma: $[H_{ij}] \in \text{Im } \varphi$

Lemma: $[H_{ij}] = \pi_* \underbrace{c_1(\xi_i \otimes \xi_j)}_{\in MU^*(H_{ij})} = MU^{2i+2j-2}_*(pt)$

(via some sort of pushforward on cohomology).

$$\begin{aligned} \text{Then } H_{ij} &= \pi_* (F(c_1(\xi_i), c_1(\xi_j))) \\ &= \pi_* \left(\sum_{n,m} \varphi(a_{nm}) c_1(\xi_i)^n c_1(\xi_j)^m \right) \end{aligned}$$

$$= \sum_{n,m} \varphi(a_{nm}) \pi_* (c_1(\xi_i)^n c_1(\xi_j)^m)$$

so need only show $\pi_*(-)$ in $\text{Im } \varphi$.

Note $\pi_*: MU^*(H_{ij}) \rightarrow MU^*(pt)$.

• geom. def'n. $Y \rightarrow X$ or oriented map given by postcomp

$$\begin{array}{ccc} Z & & \\ \downarrow f & & \\ Y & \longrightarrow & X \end{array}$$

• changes dim. b/c $\dim f = \dim Y - \dim Z$

$$\dim Y - \dim Z$$

• not multiplicative, but is an MU_* -module map.