Lecture 3: MU and complex cobordism

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Definition 1. Let X be a space and M_1, M_2 two almost complex n-manifolds. (This means that we have chosen a reduction of the structure group of the stable tangent bundle to the unitary group.) Let $f_1 : M_1 \to X$, $f_2 : M_2 \to X$ be two continuous functions. Then f_1 and f_2 are **bordant** if there exists an almost complex $(n+1)$-manifold W with $\partial W = M_1 \sqcup M_2$ and a map $H : W \to X$ such that $H|_{\partial W} = f_1 \sqcup f_2$.

We define $MU_n(X)$ to be the set of maps from n-manifolds into X modulo bordism. [Evidently this is an equivalence relation: a cylinder gives reflexivity, symmetry is clear, and gluing bordisms gives transitivity.]

$MU_*(X)$ has a graded ring structure, where addition is given by disjoint union (or equivalently, connect sum), and multiplication is given by Cartesian product.

The most obvious case to consider is when $X = *$. Then we can ignore the maps, and two manifolds are bordant iff they are the boundary of a manifold, which is just the ordinary bordism relation. We have calculated $MU_*(*) = MU_*$.

Theorem 2 (Thom-Pontryagin, Milnor)**.** *$MU_* \otimes \mathbb{Q}$ is a polynomial ring generated by the classes of $\mathbb{C}P^n$ for $n \geq 1$. In fact, $MU_* = \mathbb{Z}[t_n : |t_n| = 2n]$.*

We can choose t_n such that $(n+1)t_n = [\mathbb{C}P^n]$. Note that this ring is concentrated in even degrees: everything odd is nilbordant!

Now, recall that he Lazard ring L is isomorphic to $\mathbb{Z}[x_n : |x_n| = 2n]$. There's no canonical isomorphism $L \cong MU_*$, but choosing an orientation for MU_* gives you one. Let's go into this in more detail.

Definition 3. A multiplicative cohomology theory E^* is **complex orientable** if for $i^* : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$, $i^* : E^2\mathbb{C}P^\infty \to E^2\mathbb{C}P^1$ is surjective. Equivalently, for every complex n-plane bundle V, we can choose a Thom class $u \in E^{2n}(V, V_0)$ such that the image of u in $\pi_* E$ under the Thom isomorphism is a generator (where V_0 is the zero section). A choice of such a generator and a Thom class for all V getting sent to that generator is a **complex orientation**.

We briefly define the cohomology theory MU^* on k-manifolds X. For Z a $(k-n)$-manifold, a **complex-oriented map** $Z \to X$ is a proper map $Z \to X$ with an almost complex structure on its stable normal bundle. $MU^n(X)$ is the set of complex-oriented maps from $(k-n)$-manifolds to X modulo the **cobordism** relation: two maps $Z_1 \to X$, $Z_2 \to X$ are cobordant if there is an $(k-n+1)$-manifold W and a map $W \to X \times \mathbb{R}$ such that $Z_1 \to X$ and $Z_2 \to X$ are respectively the fibers over 0 and 1.

It is now easy to see that MU^* is complex orientable. [Indeed, a generator θ of $MU^2\mathbb{C}P^1$ is given by any map $* \to \mathbb{C}P^1$, which is equivalently a complex line in $\mathbb{C}^2$; extending this successively to a hyperplane in each $\mathbb{C}P^n$ by throwing in the new basis vectors gives an element of $MU^2\mathbb{C}P^\infty$ that evidently pulls back to θ.]

Theorem 4 (Quillen, [2])**.** *Every choice of orientation of MU_* corresponds to a formal group law over MU_*, which in turn corresponds to an isomorphism $\phi : L \xrightarrow{\sim} MU_*$.*

Definition 5. For $F = F(x,y)$ a formal group law over a torsion-free ring R, the **logarithm** of F is

$$\log_F(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t,0)} \in \mathbb{Q} \otimes R[[x]].$$

Proposition 6. $\log_F(x +_F y) = \log_F(x) + \log_F(y)$.

This can be proved by messing around with power series.

Proposition 7. *There is an orientation on MU such that*

$$\log_F(x) = \sum_{n \geq 0} \frac{[\mathbb{C}P^n]}{n+1} x^{n+1}.$$

Proof of Quillen's theorem. Let F_u be the universal formal group law over L, F the above formal group law over MU_*, $\phi : L \to MU_*$ the map classifying this FGL. We want to show that ϕ is an isomorphism. (It is a lemma that there is an isomorphism of FGLs between any two FGL over MU_*, so it suffices to prove Quillen's theorem for the single case where F is the orientation given by Proposition 7.)

First, we show $\phi \otimes \mathbb{Q}$ is an isomorphism. $F_u \otimes \mathbb{Q}$ has logarithm $\sum_{n \geq 0} \frac{p_n}{n+1} x^{n+1}$ for some p_n, and $F \otimes \mathbb{Q}$ has logarithm $\sum_{n \geq 0} \frac{[\mathbb{C}P^n]}{n+1} x^{n+1}$. Thus the map must send p_n to $[\mathbb{C}P^n]$, meaning that the degree of p_n is $2n$, and it is then clear that the p_n are generators of L.

Second, we show that ϕ is surjective. Consider the Milnor hypersuraces $H_{ij} \subseteq \mathbb{C}P^i \times \mathbb{C}P^j$ for $i \leq j$, which are cut out by the respective equations $x_0 y_0 + \cdots + x_i y_i = 0$. We claim that these are in the image of ϕ and they generate MU_*. Indeed, let $f_i : H_{ij} \to \mathbb{C}P^i, H_{ij} \to \mathbb{C}P^j$ be given by inclusion into $\mathbb{C}P^i \times \mathbb{C}P^j$ followed by projection, and let ξ_i and ξ_j be the pullbacks of the tautological line bundles along f_i and f_j respectively. Then $\xi_i \otimes \xi_j$ is a line bundle on H_{ij} which is classified by some map $f : H_{ij} \to \mathbb{C}P^\infty$. Let $\pi : H_{ij} \to *$.

There are now three substeps. First, there is a covariant 'pushforward' map $\pi_* : MU^*(H_{ij}) \to MU^*$ such that $\pi_* c_1^{MU^*}(\xi_i \otimes \xi_j) = [H_{ij}]$. If have a map $f : Y \to X$ and view $MU^*(Y)$ as the set of complex-oriented maps $Z \to Y$ modulo cobordism, then this pushforward f_* is defined by simply postcomposing these maps with f. Note that this raises the degree by $\dim X - \dim Y$, and it is not a ring homomorphism, though it is an MU^*-module homomorphism, as $Y \to X$ is a map 'over the point.'

Second, $[H_{ij}]$ is in the image of ϕ. We have

$$\begin{aligned}
[H_{ij}] &= \pi_*(F(c_1(\xi_i), c_1(\xi_j))) \\
&= \pi_* \left(\sum_{n,m} \phi(a_{nm}) c_1(\xi_1)^n c_1(\xi_j)^m \right) \\
&= \sum_{n,m} \phi(a_{nm}) \pi_*(c_1(\xi_1)^n c_1(\xi_j)^m) \\
&= \sum_{n,m} \phi(a_{nm}) [\mathbb{C}P^{n-i}][\mathbb{C}P^{n-j}].
\end{aligned}$$

Third, the classes $[H_{ij}]$ generate MU_*. To do this one must develop a theory of Chern numbers [1].

Finally, we have a map $L \to MU_*$ which is surjective and an isomorphism when tensored with $\mathbb{Q}$, and L is torsion free. Therefore, this is an isomorphism. □

We conclude by constructing the spectrum MU. $MU(n)$ is the Thom space of the tautological bundle γ_n over $BU(n)$. If $i : BU(n) \to BU(n+1)$ is the standard inclusion, then $i^* \gamma_{n+1} \cong \gamma_n \oplus 1$ has Thom space $\Sigma^2 MU(n)$. ('These are the complex 1's.') Thus i induces a map $\Sigma^2 MU(n) \to MU(n+1)$, and these define a spectrum (for example, we can let $(MU)_{2n} = MU(n)$, $(MU)_{2n+1} = \Sigma MU(n)$). By the Thom-Pontryagin theorem, this is in fact the desired spectrum MU.

Bibliography

[1] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies **76**, Princeton University Press, Princeton, 1974.

[2] D. Quillen, 'On the formal group laws of unoriented and complex cobordism theory,' *Bulletin of the American Mathematical Society* **75** (6), 1293–1298.