

$$E^0(X) := \pi_0(X, E) \quad \pi_0(X, S) = \pi_0 \text{hom}(E^* S, X, S)$$

$$E^* := E^*(S) = \pi_0 \text{hom}(X, S)$$

Johan Kuster

$$MU, MU^*, MU^*(-), MU_*(-)$$

S!

$$L \cong \mathbb{Z}[a_i] / \sim \cong \mathbb{Z}[x_i] \quad |x_i| = d_i$$

Given any spectrum E , a complex orientation is a choice of Thom class $\nu_V \in \tilde{E}^{2n}(Th(V))$ $\forall$ $\mathbb{C}$ bundles $V \rightarrow X$ satisfying naturality, etc. This is equivalent to choosing an iso

$$E^* \mathbb{C}P^\infty \cong E^*[[x]]$$

$\hookrightarrow$ generator for topological bundle

This gives a formal group law by examining $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ so we get a map $L \rightarrow E^*$ of rings.

This turns out to be the same as a multiplicative map $MU \rightarrow E$.

Remark $MU^*(pt) = MU^* = MU_{-*}$. Why?

We have a functor $\bar{L} = \text{Hom}(L, -) : \text{Rings} \rightarrow \text{Sets}$
 $R \mapsto \{ \text{FGL's over } R \}$

We want $\bar{L}$ to be a groupoid-valued functor, not just Sets-valued.

Algebra

Categories

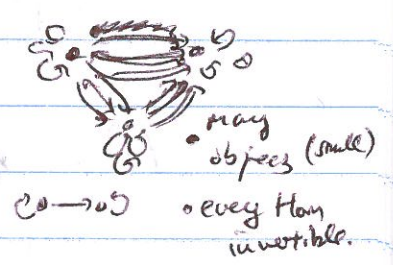
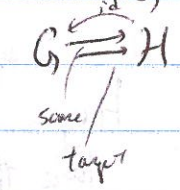
Group

$$G, G \times G \rightarrow G, \text{id}, \text{inv.}$$

one object, invertible homs

Groupoid

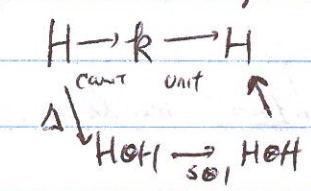
Can't always compare 2 elements a, b .
 But when a, b are in same connected component, $(abc) = a(bc)$.



Hopf Alg

base field k .
 Bialgebra H . (an algebra and a coalgebra.)

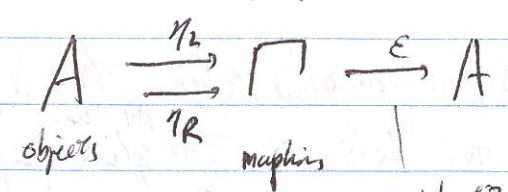
cogroup object in $k\text{-alg}$.



If H is commutative as a ring (not necessarily cocommutative), then H is a group object in affine schemes. i.e., naturally a functor $\text{Rings} \rightarrow \text{Groups}$ $\text{Hom}(H, -)$.
 (i.e., $\text{Spec}(H)$ is a group.)

antipode map $H \xrightarrow{S} H$

Hopf algebra is a cogroupoid object in rings
 So



η_1, η_2 : comultiplication, comultiplication

coidentity

(A, Γ) are rings.

antipode $\Gamma \rightarrow \Gamma$
 $\Delta: \Gamma \rightarrow \Gamma \otimes \Gamma$ comult.
~~comultiplication~~

Satisfying

$$(1) \begin{array}{ccc} A & \xrightarrow{\eta_L} & \Gamma \\ & \searrow \eta_R & \downarrow \varepsilon \\ & & A \end{array}$$

$$(2) \begin{array}{ccc} \Gamma & \xrightarrow{\Delta} & \Gamma \otimes_A \Gamma \\ & \searrow & \downarrow \varepsilon \otimes \text{id} \\ & & \Gamma \end{array}$$

$$(3) \begin{array}{ccc} A & \xrightarrow{\eta_L} & \Gamma \\ & \searrow \eta_R & \downarrow c \\ & & \Gamma \end{array}$$

Haynes Miller inverted these things for a reason

$$(4) c^2 = \text{id}$$

Note $\Gamma \otimes_A \Gamma$ has two different module structures to define $\otimes_A$

$$A \xrightarrow{\eta_L} \Gamma, A \xrightarrow{\eta_R} \Gamma \text{ define two diff module structures}$$

Applying $\text{hom}(-, R)$ on ~~the~~ the diagrams $A \rightrightarrows \Gamma$

we get a quiver $\text{hom}(A, R) \rightrightarrows \text{hom}(\Gamma, R)$

You can think of such a thing as an affine scheme that happens to take values in "Groups" not "sets." ~~But~~ ^{All the} the group properties carry over but for the strict category of groupoids. This is not really a stack, it's an honest fiber $\text{Rings} \rightarrow \text{Groups}$.

Example Formal Group Laws

$$Q \in F^*(\mathbb{Z}) \quad Q = \sum a_i x^i, a_0 = \text{unit}$$

If $a_0 = 1$, this is called a strict isomorphism

Ring of strict isomorphisms $= W = \mathbb{L}[a_1, a_2, \dots]$

$$\mathbb{L} \xrightarrow[\eta_R]{\eta_L} W \quad \eta_R = \eta_L c$$

$\eta_L = \text{inclusions}$. Define formal group law over W .

This gives new formal group law given by obvious polynomial; η_R classifies that. Good luck writing this down

Ex E a ring spectrum. We have $E \xrightarrow{\eta_L} E \otimes E \xrightarrow{\text{switch}} E \otimes E$

We want to say that

$$E \otimes E \xrightarrow[\eta_R]{\eta_L} E \otimes E$$

Almost never an iso:

$$\pi_* A \otimes \pi_* B \rightarrow \pi_* (A \wedge B)$$

is also a Hapt algebra. This is only true w/ some hypotheses.

If you assume η_L is flat (ie, $E \otimes E$ is flat as E) you can

$$\text{We want to say } E \otimes E \otimes E \otimes X \xrightarrow{\eta_L} \pi_* (E \wedge E \wedge X)$$

but we'll plug in $E = X$ to get maps $\Gamma \rightarrow \Gamma \otimes_A \Gamma$

$\pi_*(E \otimes \pi_*(E \wedge X)) = \pi_*(E \wedge X)$
 $\downarrow$ $\pi_*(E) \otimes \pi_*(E \wedge X)$ Yes $E \wedge E \wedge X \rightarrow E \wedge X$
 $\pi_*(E \wedge E \wedge X) \rightarrow \pi_*(E \wedge X)$

(*) is given by

$$\pi_*(E \wedge E) \otimes_{\pi_*(E)} \pi_*(E \wedge X)$$

$\downarrow$ using mult $E \wedge E \rightarrow E$

$$\pi_*(E \wedge (E \wedge E) \wedge X)$$

$\downarrow$

$$\pi_*(E \wedge E \wedge X)$$

This composite is NOT always an isomorphism, but it is if $E \wedge E$ is flat over E via the universal coefficient theorem.

Examine

$$\begin{array}{ccc} E \wedge E \otimes_{E \wedge E} E \wedge X & \xrightarrow{\sim} & E \wedge E \otimes_{E \wedge E} E \wedge X \xrightarrow{\pi_*} E \wedge E \otimes_{E \wedge E} E \wedge X \\ \downarrow & & \downarrow \text{iso?} \\ \pi_*(E \wedge E \wedge E \wedge X) & \xrightarrow{\sim} & \pi_*(E \wedge E \wedge E \wedge X) \xrightarrow{\pi_*} \pi_*(E \wedge E \wedge X) \end{array}$$

$\xrightarrow{\sim}$ are multiplication of $\bigoplus_m E \wedge E \wedge E$. $\exists$ a map by rest of diagram.

Note if $X=S$, H_0 is automatically true. If $E \wedge E$ is flat, both sides of $\downarrow$ are homology theories, and take $\mathbb{Z}$ to $\mathbb{Q}$. So we're done.

BP is E_1 . People may expect it's not E_∞ .
 Some of these spectra are E , but π_0 commutative. Strange.

Ex $MU \wedge MU$ We know $H\mathbb{Z} \wedge MU$.

Use Atiyah-Hirzebruch to compute

$$E \wedge MU = E \wedge [\beta_1, \beta_2, \dots] \quad |\beta_i| = 2i$$

$$\text{So } MU \wedge MU = MU \wedge [\beta_1, \beta_2, \dots]$$

$$\text{ie, } (L, W) \cong (MU, MU \wedge MU)$$

Comodules over Hopf Algebras

- an A -module M
- a coaction $M \rightarrow \Gamma \otimes_A M$ s.t.

$$M \rightarrow \Gamma \otimes_A M \rightarrow M$$

$\searrow \quad \nearrow$
id

and coassociativity

$$M \rightarrow \Gamma \otimes_A M \xrightarrow{\sim} \Gamma \otimes_A \Gamma \otimes_A M$$

In the stack world, this will be like a quasi-coherent sheaf on the stack given by (A, Γ) .

Facts (1) If η_L is flat, $\exists$ enough injectives in (A, Γ) -comod.

(2) Change of rings. A map $f: A \rightarrow B$ induces map $\Gamma \rightarrow \beta \otimes_A \Gamma \otimes_A B \xrightarrow{\sim} \Gamma_B$

which gives a functor (A, Γ) -comod $\rightarrow (B, \Gamma_B)$ -comod.

(3) With some hypotheses, $H^*(A, \Gamma; M) \subseteq H^*(B, \Gamma_B; B \otimes_A M)$.

What's cohomology?

$$M^c := \{m \in M \mid \Delta(m) \neq 1 \otimes m\}$$

This is a module over A^c .

$$F: M \rightarrow M^c; \text{ the derived functor of this is } H^q.$$

$$\text{Note } M^c = \text{hom}_{(A, D)\text{-mod}}(A, M)$$

The above change of rings theorem is non-trivial.

Fix a prime p . R a $\mathbb{Z}_p$ -algebra.

$$\text{Def } [n]_F(x) = \underbrace{x + x + \dots + x}_n$$

If $n \equiv 1 \pmod{p}$, this is invertible. So we can define $[n]_F$ for $n \neq 0$ as well. If n is coprime to p , define

$$f_n(x) := [n]_F \sum_{i=1}^n \xi^i x$$

F is p -typical if $f_n(x) = 0 \quad \forall (n, p) = 1$.

Mult. FGL is NOT.

FGL for elliptic curves are NOT.

In gen it's hard to determine whether a given FGL is p -typical.

Assume R torsion free. Recall $\log_F(x)$, where $\log_F(x+y) = \log_F(x) + \log_F(y)$.

$$\sum_j m_j x^{j+1}, m_0 = 1$$

Applying to $f_n(x)$,

$$0 = \log_F f_n(x) = \frac{1}{n} \sum_{i=0}^n \log_F(\xi^i x)$$

$$= \frac{1}{n} \sum_{i=0}^n \sum_j m_j \xi^{ji} x^{j+1}$$

$$\Rightarrow m_j = 0 \quad \forall j \not\equiv p^k - 1$$

$$\text{So } \log_F(x) = \sum_{p^k=1} m_{p^k} x^{p^k}$$

Now we see

$$[p]_F(x) = \sum_i u_i x^{p^i}$$

by inductively determining u_i .

If R is torsion-free, any FGL F . Then define

$$f^{-1}(x) := \sum_{p \nmid n} [n]_F f_n(x), \quad \mu(n) = \begin{cases} 0 & n \text{ div by square} \\ (-1)^r & n \text{ prod of } r \text{ distinct primes} \end{cases}$$

The p -series of F .
 p -typical FGLs are determined by their p -series.

then $\log_F(f^*(x)) = \sum_{p^i-1} x^{p^i}$

So any FFL is strictly isomorphic to a p -typical one if R is torsion-free

Ex over $L \otimes \mathbb{Z}_p$, this corresponds to $\Phi^2 L \otimes \mathbb{Z}_p$ where

$$\Phi(x_i) = \begin{cases} x_i & i = p^k - 1 \\ 0 & \text{else} \end{cases}$$

This gives the $\mathbb{Z}$ of FFLs from before. If FFL is p -typical, $\Phi^2 = \Phi$.

Def. $V = \text{image}(\Phi)$

This V classifies p -typical FFLs

It turns out $V \cong \mathbb{Z}_p[V_1, V_2, \dots]$ $|V_i| = 2(p^i - 1)$

Since V_i grow exponentially, any particular problem only requires very few of these V_i . Whereas MU's variables appear too often

Since analogues of p -typical FFLs are different, the analogue of W is also much, much smaller here. The analogue is

$$VT = V[t_1, t_2, \dots], \quad |t_i| = 2(p^i - 1)$$

Φ realises an equivalence (equivalent pictures, not isomorphic)

$$\Phi: \text{fgls} \otimes \mathbb{Z}_p \rightarrow p\text{-typical FFLs}$$

There's no geometric way to think of BP.

Localise MU @ p , write $MU_{(p)}$

$$g: MU_{(p)} \rightarrow MU_{(p)} \quad \text{s.t. } g^2 = g$$

The image of g is called BP.

We want g to be multiplicative. In genl, given

$$\begin{array}{ccc} & MU \otimes E & \\ \nearrow & \downarrow & \\ MU \otimes MU & \rightarrow & E \otimes E \\ \downarrow & & \downarrow \\ MU & \rightarrow & E \end{array} \quad \begin{array}{l} E \text{ complex vector space,} \\ E^*(MU) = \text{Hom}_{E^*}(E_* MU, E_*). \\ \text{Prop?} \end{array}$$

g defined by $MU_{(p)}^* \xrightarrow{\Phi} MU_{(p)}^*$. So in particular $V = BP^*$