

Localization

Irina's Talk.

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in algebra: $\mathbb{Z} \rightarrow \mathbb{Z}_{(p)}$ by inverting everything outside of (p)

$$\begin{array}{ccc} A & \longrightarrow & A \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)} \\ \text{ab. group} & & \leftarrow p\text{-component of } A. \end{array}$$

Something similar in homotopy.

X : CW spectrum, connective or bounded below

$$X \longrightarrow X_{(p)} \leftarrow p\text{-localization}$$

$$\overline{E}_*(X_{(p)}) = \overline{E}_*(X) \otimes \mathbb{Z}_{(p)}$$

Prop X -simply connected CW, finite type.

$$\overline{H}_*(X) = \text{torsion.}$$

1) if this torsion is prime to p , then $X_{(p)}$ is contractible.

2) if this is only p -torsion, then X is already p -local
 $X = X_{(p)}$.

3) $X = \bigvee_p X_{(p)}$. (X is finite type). (or take the product).

Assume we're in the p -local ^{homotopy} category of spectra.

Morava K-theories

Prop. $\forall p \exists$ a sequence of homology theory $K(n)$:

$$1) K(0)_* X = H_*(X; \mathbb{Q})$$

$$2) K(1)_* X = \text{"K-theory"}$$

$$2) . K(n)_* = \mathbb{F}_p[v_n^{\pm 1}] \quad |v_n| = 2p^n - 2.$$

3) $\exists$ a Künneth isomorphism.

$$K(n)_*(X \times Y) = K(n)_*(Y) \otimes_{K(n)_*} K(n)_*(X).$$

$$4) \text{ if } \overline{K(n)_*(X)} = 0 \Rightarrow \overline{K(n+1)_*(X)} = 0$$

for X -finite.

5) ...

Def = A p -local spectrum X has type n if n is the smallest n such that $K(n)_*X \neq 0$.

~~5)~~ 5) $K(n)_*$ is specified by its formal group law Γ_n .
It is p -typical with p -series

$$[p]_{\Gamma_n}(x) = v_n x^{p^n}.$$

Γ_n is called the Honda formal group law.

— — —
Where do they come from? B-Sullivan construction

$$y \in \pi_k MU$$

$$\sum^k MU \xrightarrow{y} MU \longrightarrow C(y)$$

Homology theory

$$\pi_* C(y) = MU_*/(y).$$

we can iterate this. and kill any ideal generated by a regular sequence. $(y_1, \dots, y_n)$

Ex) 1) kill $(x_1, x_2, \dots, x_n)$

(2)

gives $C(x_1, \dots, x_n, \dots) = H_*$

2) BP_* is a result of such construction.

$A \hookrightarrow \pi_* MU \rightarrow \pi_* BP_*$
 "kernel."
 obtained by killing A

3) Start with BP_* kill $(p, v_1, \dots, v_{n-1}, v_{n+1}, \dots) \rightarrow K(n)_*$

and $K(n) = \varinjlim \left\{ \Sigma^{-2i(p^n-1)} K(n) \xrightarrow{v_n} K(n) \right\}$.

$K(n)_*$ detects periodic maps.

Defn A map $f: \Sigma^q X \rightarrow X$ is called a self map
 It is called nilpotent if there is a t such that

$$f^t: \Sigma^{tq} X \rightarrow \Sigma^{2q} X \rightarrow \Sigma^q X \xrightarrow{f} X$$

some suspension of f^t is null-homotopic.

otherwise it is called periodic.

Thm (Periodicity, HSD)

$X, \mathbb{F}_p$ -local a type n spectrum, there exists

a self map $f: \Sigma^{d+i} X \xrightarrow{f} \Sigma^i X$ for some i

s.t. $K(n)_* f$ is an isomorphism. and $K(m)_* f = 0$ for $m > n$.

such an f is called a v_n self map.

~~Such an f is called~~

Side note.

Question: Find a functor from

Spaces $\xrightarrow{F}$ some
algebraic
category

which is ¹⁾ reasonably easy to compute and

$$2) F(f)=0 \Leftrightarrow f \sim 0$$

Fact: It's ~~reasonably~~
impossible

Question

Finite
CW-spectrum $\rightarrow$ some
alg cat.

Conj. Π_*^S is such a functor (But not
easy to compute)

Thm (Nilpotence) (D.H.S.).
of finite CW-spectrum.

A self map f is ~~stably~~ nilpotent if

some iterate of $\overline{mU}_*(f)$ is trivial

$\Downarrow$ p-local.

... $K(h)_* f$.

n	Self-map	Map on $K(n)_*$	Periodic families in ANSS
0	$S^k \xrightarrow{p} S^k \rightarrow V(0)_k$ $\Sigma^k V(0)$	p	$S^k \xrightarrow{p} S^k \xrightarrow{p} S^k \xrightarrow{p} \dots$
1	$\Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k \rightarrow V(1)_k$ Adams $q=2p-2$ $p=\text{odd}$ $q=4$ $p=2$	v_1 v_1^4	$S^{q+k} \rightarrow \Sigma^q V(0)_k \xrightarrow{\alpha^t} V(0)_k \xrightarrow{j} S^k$ $\alpha_t \in \pi_{q_t-1}$ $p=\text{odd}$ α_{4t} $p=2$ this composite is essential.
2	$\Sigma^{2(p^2-1)} V(1)_k \xrightarrow{\beta} V(1)_k \rightarrow V(2)_k$ $p \geq 5$ Smith-Toda.	multiplication by v_2 .	$S^{2t(p^2-1)} \rightarrow S^{k+2}$ "β family" Smith proved these are non-trivial
3	$\Sigma^{2(p^2-1)} V(2)_k \xrightarrow{\gamma} V(2)_k$ $p \geq 7$	v_3	Miller Ravenel Wilson proved γ_t is essential.

why is there α ?

$$\exists \alpha \in \pi_{2p-3} S \text{ s.t. } p\alpha = 0. \quad q = 2p-2.$$

$$\begin{array}{ccccc}
 \Sigma^q S^k & \xrightarrow{p} & \Sigma^q S^k & \longrightarrow & \Sigma^q V(0)_k \\
 & & \downarrow \alpha' & \swarrow \xi & \downarrow \alpha \\
 S^{k-1} & \longleftarrow & S^{k-1} & \longleftarrow & V(0)_k
 \end{array}$$

$p\alpha = 0 \Rightarrow \xi$
 $p\xi = 0$

$$\exists \beta \alpha \Sigma^q V(0)_k \rightarrow V(0)_k$$

Localization, again.

E-spectrum = homology theory

Want to associate to every spectrum X the part
of X that E can see: LEX

Ex if $X \wedge E \sim *$ want $LEX \sim *$

if $X \rightarrow Y$ induces an E equivalence

$$X \wedge E \rightarrow Y \wedge E$$

want $LEX \cong LEY$.

Bousfield Localization: X is E -local if

$\forall$ E -equivalence $S \rightarrow T$ the map.

$[T, X] \rightarrow [S, X]$ is an ~~isomorphism~~.

isomorphism.

A spectrum Y with map $X \rightarrow Y$
is an E -localization if

1) Y is E -local

2) $X \rightarrow Y$ is an E equivalence.

Construction: $LEX = \text{hocolim}_Y$

$$X \rightarrow Y$$

E -equivalence.

in algebra: invert elements in localizing set.

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in homotopy: invert the E -equivalences

localization with respect to E

Thm. (Bousfield) The category of spectra has a model structure

1) cofibrations = cof. of spaces.

2) weak equivalences are E -equivalences.

Ex. 1) $E = M(\mathbb{Z}(p))$ - Moore space

$L_E X \cong X_{(p)}$ - p -localization.

$$L_E X = X \wedge L_E S^0 \quad \text{smash localization}$$

2). $E = H\mathbb{Q}$

$$L_E X = X \wedge L_{\mathbb{Q}} S^0$$

How do we reconstruct a spectrum?

Prop. E, F, X spectra

$$E_*(L_F(X)) = 0$$

Then there exists a homotopy pull-back square

$$L_{E \vee F} X \longrightarrow L_E X$$

$$\downarrow \text{ p.b. } \downarrow$$

$$L_{F X} \longrightarrow L_F L_E X$$

if P is the pull-back

1) P is E and F local

2) $X \rightarrow P$ is E and F equivalence.

~~Cor.~~ We can localize at $K(n)$ $L_{K(n)}$
can localize at $E(n)$

$$L_{E(n)} \simeq L_{K(1) \vee K(2) \vee \dots \vee K(n)} = L_n.$$

Then. there are maps $L_{n+1} \rightarrow L_n.$

Cor.

$$L_{K(1) \vee K(2)} X \longrightarrow L_{K(2)} X$$

$$\downarrow \text{ p.b. } \downarrow$$

$$L_{K(1)} X \longrightarrow L_{K(1)} L_{K(2)} X$$

pf check $K(2)_* (L_{K(1)} X) = 0.$

Cor

$$L_n X \longrightarrow L_{K(n)} X$$

$$\downarrow \text{ p.b. } \downarrow$$

$$L_{n-1} X \longrightarrow L_{n-1} L_{K(n)} X$$

(5).

Defⁿ The chromatic tower.
of a p-local spectrum X is given by

$$\{ \cdots \rightarrow L_n X \rightarrow L_{n-1} X \rightarrow \cdots \rightarrow L_0 X \}$$

Thm Chromatic convergence
for X -finite.

$$X_p = \lim_{\leftarrow} L_n X$$

Thm Smash product thm.

$$L_n X \simeq X \wedge L_n S^0$$

Height

Extend $K(n)$

$$K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$$

$$u^{1-p^n} = v_n.$$

ω primitive p^n-1 root of unity.

In char p , p -series of a f.g.l $\mathbb{F}$
has leading term. ax^{p^n} .

If a is invertible, $\mathbb{F}$ has height n .

Defⁿ If $[p]_{\mathbb{F}}(x) = ax^{p^n} + \text{h.o.t.}$

n has height n if a is invertible.

act) $[p]_{\mathbb{F}}(x) = 0$ then $\mathbb{F}$ has height ∞ .

$$1) [P]_{F_n}(x) = \underbrace{x + \dots + x}_p = 0 \quad \text{over } \mathbb{F}_p.$$

$$2) [P]_{F_n}(x) = (1+x)^p - 1 = x^p \quad \text{height } p \text{ over } \mathbb{F}_p.$$

Thm Lazard

Two formal group laws ~~are~~ ~~iso~~ over an alg closure of $\mathbb{F}_p$ are iso $\Leftrightarrow$ they have the same height.

$X(n)_x$ has formal group law

$$[P]_{\Gamma_n}(x) = v_n x^{p^n} \quad \text{has height } n.$$

$S_n = \underline{\text{Automorphisms of } \Gamma_n}$. An automorphism is a power series $f(x)$ s.t. $f(\Gamma_n(x, y)) = \Gamma_n(f(x), f(y))$ and $f(x) = ax + \dots$ a invertible.

Because Γ_n is p -typical, it's enough to check for the p -series.

$$* \quad f([P]_{\Gamma_n}(x)) = v_n (f(x))^{p^n}$$

1) Let ω be a primitive (p^n-1) root of unity

(6)

~~is~~ $\omega(x) = \omega x$ ~~is~~ is satisfied.

2) $S: S(x) = \omega^{1+p} x^p$

$S^n(x) = S \circ \dots \circ S(x) = \omega^n x^{p^n}$ ~~is~~ is satisfied.

3). $(\mathbb{Z}_p[\omega])^* \subseteq \mathbb{S}_n$.

4) $Sa = a^\sigma S$ for $\sigma = \text{Frobenius}$.

$\mathbb{S}_n = \text{Aut}(\Gamma_n) = \left(\mathbb{Z}_p[\omega] \langle S \rangle / (S^n = p, aS = Sa^\sigma) \right)^*$

$\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $\mathbb{S}_n$ ~~on~~

$G_n = \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$

$\mathbb{S}_n$: Morava stabilizer group

G_n : Big or extended Morava stabilizer group.

$\frac{H_{\text{Morava}}}{G_n}$ acts on Morava E-theory $(E_n)_*$ and there is a spectral sequence $H^*(G_n, (E_n)_*) \Rightarrow \pi_* L_{K(n)} S^0$

