

Irina Bobkova: Height Localization:

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in algebra: $\mathbb{Z} \rightarrow \mathbb{Z}_{(p)}$

$A \xrightarrow{\text{abgrp}} A \otimes \mathbb{Z}_{(p)}$ p component of A

in htpy theory: $X \xrightarrow{\text{(connective) CW-sp}} X_{(p)}$ localization.

$$E_*(X_{(p)}) = \bar{E}_*(X) \otimes \mathbb{Z}_{(p)}$$

Prop: X simply conn. CW. $\bar{H}_*(X)$ consists of torsion

- 1.) if this torsion is prime to p , $X_{(p)}$ is contractible
- 2.) if only p -torsion, $X \simeq X_{(p)}$; X is p -local
- 3.) $X \simeq \bigvee_p X_{(p)}$ (for finite type...) (in general: $X \cong \prod_p X_{(p)}$)

• From now on we live in p -local world • (fixed p)

Morava K-theories

Prop: $\forall p$ $\exists$ a sequence of homology theories $K(n)$ s.t.

$$1.) K(0)_*(X) = H_*(X; \mathbb{Q})$$

$K(1)_*(X)$ related to K -theory

$$2.) K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}] \quad |v_n| = 2p^n - 2$$

3.) $\exists$ Künneth isom.

$$K(n)_*(X \times Y) = K(n)_*(X) \otimes K(n)_*(Y)$$

$$4.) \text{ if } \overline{K(n)}_*(X) = 0 \Rightarrow \overline{K(n-1)}_*(X) = 0 \text{ (if } X \text{ finite)}$$

5.) they have p -typical FGL w/ p -series $[p]_*(x) = v_n x^{p^n}$ for $K(n)$.

Defn: A p -local CW-spectrum X if n is the smallest s.t. $K(n)_* X \neq 0$.

Defn: The p -FGL of $K(n)$ is the Honda formal grouplaw: Γ_n with $[p]_{\Gamma_n}(x) = v_n x^{p^n}$.

Where do the $K(n)_*$ come from?

$$Y \in \pi_* MU \quad \Sigma^k MU \xrightarrow{Y} MU \rightarrow C(Y) \xrightarrow{\text{homology theory}} \pi_*(C(Y)) = MU$$

(Bass-Sullivan construction)

can iterate to kill any ideal gen. by a reg. sequence.

If $(y_1, \dots, y_n)$ is a regular sequence, get $C(y_1, \dots, y_n)$

Ex: kill $(x_1, \dots, x_n)$ $C(x_1, \dots, x_n) = H_*$

Ex: BP_* is a result of such construction (mostly)

$$A_* \hookrightarrow MU_* \rightarrow \pi_* BP$$

Ex: start w/ BP_* . kill $(p, v_1, \dots, v_n, v_{n+1}, v_{n+2}, \dots)$ to get $K(n)_*$ (conn. version)

$$K(n) = \varinjlim \Sigma^{-2i(p^n-1)} K(n)$$

• $K(n)_*$ detects periodic maps.

Defn: $f: S^d X \rightarrow X$ is called a self-map. It is nilpotent if some ^{suspension of some} iteration $\Sigma^d X \xrightarrow{f} \Sigma^{d+1} X \xrightarrow{f} \dots \Sigma^d X \xrightarrow{f} X$

f is null homotopic. Otherwise it is periodic

Thm (Periodicity DHS)

X p -local, finite, type n . $\exists$ selfmap $f: \Sigma^{d+i} X \rightarrow \Sigma^i X$ for some i, d s.t. $K(m)_* f$ is an isom and $K(m)_* f = 0$ for $m > n$. Such f is called a V_n -map.

Question: find a functor spaces $\bar{F} \rightarrow$ some alg. cat. which is 1) reasonably easy to compute 2) $F(f) = 0 \iff f \sim 0$ impossible

ex: change spaces to fin. spectra, then conjecturally π_* by Freyd's generating Hypothesis.

instead, we get:

Nilpotence Thm (DHS) A self map f (of fin (Wsp)) is nilpotent $\iff$ some iterate of $MU_*(f)$ is trivial
 $\iff$ $\overset{p\text{-local}}{K(n)_*(f)}$ triv.

cofiber.

n	self map	Map on $K(n)_*$	periodic form in ANSS
0	$S^k \xrightarrow{p} S^k \rightarrow V(0)_k = \Sigma^k V(0)_k$	mult. by p	$S^k \xrightarrow{p} S^k \xrightarrow{p} S^k$
1	$\Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k \rightarrow V(1)_k$ Assums: $q = 2p-2$ p odd $p=2$	mult. by v_1 $p=2: v_1^4$	$S^{q+k} \xrightarrow{\alpha} \Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k \xrightarrow{1} S^{k+1}$ $\alpha \in \Pi_{q+k-1}^{p \text{ odd}}$ is essential (Adams) $\alpha \in \Pi_{q+k-1}^{p=2}$
2	$\Sigma^{2(p^2-1)} V(1)_k \xrightarrow{\beta} V(1)_k \rightarrow V(2)_k$ $p \geq 5$ Smith-Toda $\rightarrow V(2)_k$	mult. by v_2	$S^{2(p^2-1)+k} \xrightarrow{\beta} S^{k+2p}$ Smith is essential
3	$\Sigma^q V(2)_k \xrightarrow{\gamma} V(2)_k$ $q = 2(p^3-1)$ Miller...	mult. by v_3	Miller-Ravenel-Wilson - corresp. γ^+ family is essential

There is a potential obstruction at all primes for v_k that means that this process probably stops: so we don't expect mult by v_k , only by a power of v_k .

Why is there α ?

$$\exists \alpha' \in \Pi_{2p-3} S \text{ s.t. } p\alpha' = 0.$$

$$q = 2p-2.$$

$$\Sigma^q S^k \xrightarrow{p} \Sigma^q S^k \rightarrow \Sigma^q V(0)_k$$

$$\begin{array}{ccc} & \downarrow \alpha & \downarrow \alpha \\ S^{k-1} & \xrightarrow{p} & S^{k-1} \leftarrow V(0)_k \end{array}$$

α also p -torsion, so get $\alpha: \Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k$
(NB at $p=2$, $p \neq 2$; Moore sp. at identity has order 4.)

Localization again

E a spectrum.

Want to associate to each spectrum X the part of X that E can see: $L_E X$ the localization of X

ex: if $X \wedge E \sim *$ want $L_E X \sim *$

if $X \rightarrow Y$ induces equiv $X \wedge E \rightarrow Y \wedge E$ want
 $L_E X = L_E Y$

Defn: X is E_* local $\Leftrightarrow \forall E$ -equiv. $S \rightarrow T$ the map.
 $[T, X] \rightarrow [S, X]$ is an iso.

A spectrum Y with a map $X \rightarrow Y$ is called the E -local.
 of X if. 1.) Y is E -local 2.) $X \rightarrow Y$ is an E -equiv

construction: $L_E X = \operatorname{hocolim}_{\substack{X \rightarrow Y \\ E\text{-equiv}}} Y$ (its a thm that this works.)

In algebra: invert elts. in localizing set

in htpy cat: invert the E -equiv to local w.r.t. E .

Thm (Bousfield) The cat. of spectra has a model structure.

1.) cofib. are the usual cofibrations.

2.) we. are E -equiv.

Ex: 1.) $E = M(\mathbb{Z}/p)$ moore space. $L_E X = X_{(p)}$ p -local

Property: $X \wedge L_E S^0 = * \Leftrightarrow L_E X$ "smashing localization"

2.) $E = H\mathbb{Q}$. $L_E X = X \wedge L_{\mathbb{Q}} S$ also smashing

Reconstructing a spectrum:

Prop: E, F, X spectra, $E_*(L_F X) = 0$

Then $\exists$ a htpy pullback square

$$\begin{array}{ccc} L_{E \vee F} X & \longrightarrow & L_E X \\ \downarrow & & \downarrow \\ L_F X & \longrightarrow & L_E L_E X \end{array}$$

Pf: Check pullback is
 $E \vee F$ local. +
 $X \rightarrow P$ is $E \vee F$
 equiv.

We can localize at $K(n)$ to get $L_{K(n)}$, or at $E(n)$

$$L_{E(n)} = L_{K(n)} \vee L_{K(n+1)} = L_n$$

$\exists$ maps $L_{n+1} \rightarrow L_n$

$$\begin{array}{ccc} \text{Cor: } L_{K(n)} \vee L_{K(n+1)} X & \longrightarrow & L_{K(n+1)} X \\ \downarrow & & \downarrow \\ L_{K(n)} X & \longrightarrow & L_{K(n)} L_{K(n+1)} X \end{array}$$

Pf: Check $K(n+1)_* (L_{K(n)} X) = 0$
(consequence of periodicity/
type n stuff)

Also holds for higher n:

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{n+1} X \\ \downarrow & & \downarrow \\ L_{n+1} X & \longrightarrow & L_{n+1} (L_{n+1} X) \end{array}$$

Defn: There is a chromatic tower of a p-local spectrum X

$$L_0 X \leftarrow L_1 X \leftarrow L_2 X \leftarrow \dots$$

Thm (Chromatic convergence): X p-local finite CW spectrum

$$X = \varprojlim L_n X$$

Thm (Smash prod thm): $L_n X \cong X \wedge L_n S^0$ ("smashing localization")

so we're reduced to studying $L_n S^0$ (Not for $L_{K(n)}$)

Height

Fact: In char. p, p-series of a FGL F has leading term ax^{p^n} , (a is a unit.)

Defn: $[p]_F(x) = ax^{p^n} + \text{h.o.t.}$ with a a unit, then n is the height of the FGL.

Ex: 1) $[p]_{F_a}(x) = x + \dots + x = 0$ is height ∞
(F_a = add. FGL)

2) $[p]_{F_n}(x) = (1+x)^p - 1 = x^p + \dots$ is height 1.
(mult. FGL)

Thm (Lazard) Two fgl over the algebraic closure of F_p are isomorphic $\iff$ they have the same height.

$K(n)_*$ has FGL Γ called the Honda FGL s.t.

$$[p]_{\Gamma}(x) = v_n x^{p^n} \quad \text{so } \text{it has height } n$$

Automorphisms of $\Gamma_n = S_n$

$$f(\Gamma_n(x, y)) = \Gamma_n(f(x), f(y)) \quad \text{power series w/ invert. 1st coeff.}$$

Enough to check this for the p-series

$$f([p]_{\Gamma_n}(x)) = v_n f(x)^{p^n} \quad *$$

1.) let ω be the primitive $(p^n - 1)$ root of unity

$$w(x) = \omega x \quad \text{gives an auto. of } \Gamma_n$$

2.) $s: S(x) = u^{1-p} x^p$ (here we've extended $K(n)$ by a root u of $x^{p^n-1} = 1$)

$$S^n(x) = s \circ \dots \circ s(x) = p^n v_n x^{p^n} \quad * \text{ satisfied}$$

3.) $s \circ \sigma = \sigma \circ s$ ($\sigma = \text{Frobenius.}$)

$$\text{Thus } (\mathbb{Z}_p[\omega])^{\times} \subset S_n \quad S \subset S_n$$

specifically: $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ originally $|v_n| = 2p^n - 2$
change to $K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$ $|u| = 2$ $v_n = u^{1-p^n}$

- doesn't change htpy, they b/c it's an étale ext.

Morava
stabilizer
group

$$S_n = \text{Aut}(\Gamma_n) = \left(\frac{\mathbb{Z}_p[\omega] \langle S \rangle}{\mathbb{Z}_p} \right)^{\times} / \langle S^n = p, uS = S\sigma \rangle^{\times}$$

$\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on S_n and $G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$
is the extended Morava stabilizer group.

Then G_n acts on Morava E-theory $(E_n)_*$
(Morava) and there is a spectral sequence

$$H^*(G_n; (E_n)_*) \Rightarrow \pi_* L_{K(n)} S^0$$

if p large enough w.r.t n , this collapses.

Hopkins-Miller: G_n acts on E_n

$$(E_n)_* = \mathbb{Z}_p[\omega][u_1, \dots, u_{n-1}][u^{\pm 1}]$$

$$E(n)_* = \mathbb{Z}_p[v_1, \dots, v_{n-1}][v_n^{\pm 1}].$$