

{ Complex oriented cohomology theories } $\longrightarrow$ { formal group laws }

$$E \longmapsto F_E \in E_*[[x, y]]$$

$$\text{Determined by } e(I_1 \otimes I_2) = F_E(e(I_1), e(I_2))$$

Example 1: $HR = H(-, R)$

$$\text{coefficients} = R \quad F(x, y) = x + y$$

$$2) KU \longmapsto \mathbb{Z}[\beta, \beta^{-1}] \quad , F(x, y) = x + y + \beta x y.$$

$$3) MU \longmapsto L \text{ , universal formal group law. (Quillen's thm)}$$

$$4) BP \longmapsto V \cong \mathbb{Z}_p[v_1, v_2, \dots] \text{ universal p-typical formal gp law.}$$

$$5) E(n) \longmapsto E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n][v_n^{-1}]$$

$$6) K(n) \longmapsto \mathbb{F}_p[v_n^{\pm 1}] \text{ Honda Formal group law.}$$

$$7) E_n \longmapsto \mathbb{Z}_p[v_1, \dots, v_{n-1}][v_n^{\pm 1}] \text{ , Lubin-Tate Formal group law.}$$

universal deformation of the
Honda formal group law.

Question Given a graded formal group law F , can we
find a complex oriented cohomology theory $E(F)$ such
that $F_{E(F)} \cong F$. ?

By Quillen's thm, this is F equivalent to a commutative
 MU_* algebra? Can you lift it to a homology theory?

Answer: No in general.

we can try to define ~~$E(F)_*(X) = MU_*(X) \otimes_{MU_*} R_*$~~
 $E(F)_*(X) = MU_*(X) \otimes_{MU_*} R_*$
 where the map $MU_* \rightarrow R_*$ classifies the formal group
 law.

This satisfies all the axioms for a complex oriented
 homology theory except maybe excision, i.e.

~~excision~~

if $X \rightarrow Y \rightarrow Z$ is a cofiber sequence, we need
 an exact sequence

$$E_*(X) \rightarrow E_*(Y) \rightarrow E_*(Z)$$

This may not hold since $\otimes_{MU_*}^X$ is not exact.

If R_* is flat over MU_* , then $\otimes$ is exact
 and excision holds.

But $MU_* \cong \mathbb{Z}[x_1, x_2, \dots]$. so flat modules don't detect
 anything more.

Observation :

$MM_*(X)$ is not just any MM_* module.

This module has the structure of a comodule over the Hopf algebra. $MM_* \otimes MM_*$

Sufficient to check that $\otimes_{MM_*}$ is exact ~~over~~ for

$(MM_*, MM_* \otimes MM_*)$ - comodule.

Defⁿ A graded L -module M_* is called Landweber exact if the functor

$$M_* \otimes_L - : \{(L, LB) \text{ - comodules}\} \rightarrow \{\text{Ab groups}\}$$

if this functor is exact.

$$L = \text{Lazard ring} \quad \otimes_{MM_*} \\ L = \langle b_1, b_2, \dots \rangle = MM_* \otimes MM_*$$

Summary if the graded formal group law F is Landweber exact, then $E(F)$ is a complex oriented cohomology theory.

Given a prime number p , let $v_i \in L$.

be the coefficient of x^{p^i} in the p -series of the universal formal group law.

$$[p]_{\text{univ}}(x) = \dots + v_i x^{p^i} + \dots$$

Theorem (Landweber)

A graded L -module M_* is Landweber exact
iff and only if, for every prime p .

$v_0 = p, v_1, v_2, \dots$ is a regular sequence for M .

$$\text{i.e. } M/(p, v_1, \dots, v_{n-1}) \xrightarrow{v_n} M/(p, v_1, \dots, v_n)$$

is injective for each n .

Sketch of proof: Use algebra to reduce this to
Lazard's theorem on the uniqueness of height n .

F.G.L.'s.

Examples

1) $F(x, y) = x + y$ over R_*

$$[p]_F(x) = px \quad v_0 = p. \quad v_1 = v_2 = \dots = v_{p-1} = 0.$$

$\overline{F}$ is Landweber exact iff $\mathbb{Q} \subset R_*$

$$MU_*(X) \otimes_L R_*$$

$$MU_*(X) \otimes_L R_0 = H_*(X, R_*)$$

2) $F(x, y) = x + y + uxy$ $u \in R_2.$

$$F(x, y) = [(ux + 1)(uy + 1) - 1]u^{-1} \quad \text{Formally.}$$

$$\Rightarrow \quad [p]_F(x) = [(ux + 1)^p - 1]u^{-1}$$

$$v_0 = p \quad v_1 = u^{p-1} \quad v_2 = v_3 = \dots = 0.$$

$\overline{F}$ is Landweber exact iff R_* is torsion free and u is invertible.

$$R_* = \mathbb{Z}[u, u^{-1}]$$

$$MU_*(X) \otimes_L \mathbb{Z}[u^{\pm 1}] = KU_*(X)$$

(Conner-Floyd isomorphism)

$\rightarrow$ write down the map $MU \rightarrow KU$.

3) The universal p -typical FGL over $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$ is Landweber exact.

$$MU_*(X) \otimes_{\mathbb{Z}_{(p)}[v_1, \dots]} \approx BP_*(X).$$

4) $\mathbb{Z}_{(p)}[v_1, \dots, v_n^{\pm 1}]$ is Landweber exact and

$$E(F) \approx E(h)$$

5) Honda formal group law on $\mathbb{F}_p[v_n^{\pm 1}]$ not Landweber exact because $\mathbb{F}_p[v_n^{\pm 1}]$ is not torsion free.

6) $\mathbb{Z}_p[v_1, \dots, v_n][u^{\pm 1}]$ is Landweber exact.

Marc Hovey's, Part II Elliptic cohomology.

Let $C \rightarrow \operatorname{Spec} R$ be a smooth 1-dimensional abelian group scheme.

with

$$e: \operatorname{Spec} R \rightarrow C$$

Let $\omega_C = \omega_{\text{normal sheaf of } e}$ = dual of the Lie algebra.

in general ω_C is a free R -module of rank 1.

We assume that we are given a trivialization

$$\omega_C \cong R \times$$

$$\hat{C} = \operatorname{colim} \operatorname{Spec}(\omega_C / I^n)$$

I : ideal defining e .

$\widehat{\quad}$ neighborhood of e .

$$= \operatorname{Spf}(\varprojlim \omega_C / I^n) \cong \operatorname{Spf} R[[x]]$$

Group structure induces a formal group law

$$R[[x]] \longrightarrow R[[x, y]] = R[[x]] \hat{\otimes}_R R[[x]]$$

$$x \longmapsto F(x, y).$$

$F(x, y)$ is a formal group law.

Question: For which C/R is this a Landweber exact formal group law?

Examples. There are ^{essentially (locally)} 3 examples: G_m, G_a , Elliptic curves.

• G_a $\text{Spec } R[x] \xrightarrow{\varepsilon} \text{Spec } (R)$ with multiplication.

$$R[x] \longrightarrow R[x, y] = R[x] \hat{\otimes}_R R[x]$$

$$x \longmapsto x \otimes 1 + 1 \otimes x$$

unit $R[x] \longrightarrow R$
 $x \longmapsto 0$.

We complete $R[x]$ at $0 = (x)$.

Hence this gives $R[x] \longrightarrow R[x] \hat{\otimes}_R R[x] = R[x, y]$
 $x \longmapsto x \otimes 1 + 1 \otimes x = x + y$.

$\Rightarrow \hat{G}_a$ is the additive formal group law.

• G_m : $\text{Spec } R[t, t^{-1}] \longrightarrow \text{Spec } R$
multiplication $R[t^{\pm 1}] \longrightarrow R[t^{\pm 1}] \hat{\otimes}_R R[t^{\pm 1}]$
 $t \longmapsto t \otimes t$

unit $R[t, t^{-1}] \longrightarrow R$
 $t \longmapsto 1$.

We need to complete at $1 = (t^{-1})$.

First make the change of variable.

$$x = t^{-1} \Rightarrow t = x^{-1}$$

$$R[x, \frac{1}{x+1}] \longrightarrow R[x, \frac{1}{x+1}] \otimes_R R[x, \frac{1}{x+1}]$$

$$t^{-1} = x \longmapsto t \otimes t - 1 \otimes 1 = (x+1) \otimes (x+1) - 1 \otimes 1 \\ = x \otimes x + x \otimes 1 + 1 \otimes x - 1 \otimes 1$$

$$\text{Complete at } (x) \Rightarrow R[x] \longrightarrow R[x] \hat{\otimes}_R R[x] = R[x, y] \\ x \longmapsto x+y+xy.$$

$\hat{G}_m$: the multiplicative formal group law.

$C \xrightarrow{e} \text{Spec}(R)$ is an elliptic curve.

$\omega_C \cong R$ induces a closed immersion into $\mathbb{P}_R^2$

$$C \hookrightarrow \mathbb{P}_R^2$$

C := projective closure

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

How we compute $\hat{C}$?

Recipe: • make a change of coordinates to bring e in the chart e is at infinity. in the Weierstrass form.

$$(x, y) \rightsquigarrow (z, w) \Rightarrow E = (0, 0). \quad w(z) = z^3 + \dots$$

$$\begin{aligned} P &= (z, w) \\ P' &= (z', w') \end{aligned} \Rightarrow P + Q = F(z, w, z', w')$$

~~$F(z, w, z')$~~

• ~~Express~~ Solve the equation for the curve.
and in $\mathbb{P}^2$

that expresses w as a power series in z .

$$F(z, w(z), z', w'(z')).$$

Classical Theorem / Defⁿ

Let k be a field of characteristic $p > 0$

and C an elliptic curve. Then $\hat{C}$

has height ≤ 2 . If it has height 1

then C is called an ordinary elliptic curve.

If it has height 2, C is called super-singular.

There are only finitely many super-singular curves up to isomorphism. Proof: see Silverman.

When $\hat{C}$ is Landweber-Exact?

$\rightarrow v_0 = p$, R must be torsion free.

$\rightarrow v_1 \in R/p$ must not be a zero divisor, i.e.,

The generic fiber of $C \otimes \mathbb{F}_p$ is ordinary.

$\rightarrow v_2 \in R/(p, v_1)$ is always invertible.

Example. away from $p \neq 2$ any elliptic curve is Zariski locally of the form.
 $y^2 = x(x-1)(x-\lambda)$

$\mathbb{Z}[\frac{1}{2}, \lambda][\frac{1}{\lambda(\lambda-1)}] \rightarrow$ Landweber exact.

① $\textcircled{A} \textcircled{1} m_{fg} \quad \textcircled{1} \text{Spec}(R) \xrightarrow{F} m_{fg} \hookrightarrow \text{Fgroups. over } R$

②. $\begin{array}{ccc} \text{Spec}(S) & \xrightarrow{G} & m_{fg} \\ f \downarrow & \searrow \varphi & \\ \text{Spec}(R) & \xrightarrow{F} & \end{array}$ $\begin{array}{l} g: R \rightarrow S \\ \varphi: G \xrightarrow{\sim} f^* F \end{array}$

③ $\text{Spec}(R) \xrightarrow{F} m_{fg}$ is flat $\iff F$ is Landweber exact
 LEFT is a criterion for flatness.

Suppose there was a function ϵ - (THIS CAN'T BE TRUE).

$\epsilon(\text{Spec}(R) \xrightarrow[\uparrow \text{Flat}]{F} m_{fg}) = E(R, F)$ spectrum.

then $\text{holim } \epsilon = S^0$ and the homotopy inverse limit S.S. would be the A.N.S.S.

$$\pi_k E(R, F) = \begin{cases} 0 & t \text{ odd} \\ \omega_F^{\otimes k} & t=2k. \end{cases}$$

$\cong R[u^{\pm 1}]$ if $u \in \omega_F$ is a generator

Hopkins Miller

$$m_{\text{ell}} \rightarrow \underset{\substack{\text{open} \\ \text{height } 2}}{U(2)} \subseteq m_{\text{fs}}$$

$\hookrightarrow$ For m_{ell} , F-étale sheaf is true

$$\boxed{\text{holm } \mathcal{E} = Tm f.}$$

