

Rob Legg: $K(1)$ -local sphere

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" $K(1)$ -local sphere = $1^{\pm}$ chromatic layer of S "

What does this mean? Let's find out.

I. Reference: Miller, Ravenel, Wilson, "Periodic phenomena..."

Q: Is there periodicity (algebraic) in E_2 of ANSS manifesting observed infinite periodicity (geometric)?

Consider

$$0 \rightarrow BP_* \xrightarrow{p^i} BP_* \rightarrow BP_*/p^i \rightarrow 0$$

Apply $\text{Ext} \rightsquigarrow$ LES. Elements of order p^i live in image of $\delta: \text{Ext}^0(BP_*/p^i) \rightarrow \text{Ext}^1(BP_*)$.
Take a colimit to consider all p -torsion:

$$0 \rightarrow BP_* \rightarrow \bar{p}' BP_* \rightarrow BP_*/(p^\infty) \rightarrow 0$$

Repeat with v_1 , etc. and splice to obtain chromatic resolution:

$$0 \rightarrow \bar{p}' BP_* \rightarrow v_1^{-1} BP_*/p^\infty \rightarrow \dots$$

$$\rightsquigarrow \text{s.s. } E_r^{n,s,t} = \text{Ext}^{s,t}(v_n^{-1} BP_*/I_n^\infty) \Rightarrow \text{Ext}^{n+s,t}(BP_*)$$
$$d_r: E_r^{n,s,t} \rightarrow E_r^{n+r,s-r+1,t}$$

We don't know this E, either, so let's do it again

Try to relate $\text{Ext}^*(v_n^{-1}BP_\star/I_n^\infty)$ to $\text{Ext}^*(v_n^{-1}BP_\star/I_n)$

Notation $M_n^m = v_{n+m}^{-1}BP_\star / (p, v_1, \dots, v_{n-1}, v_n^\infty, \dots, v_{n+m-1}^\infty)$

Want M_0^n . Get it from M_n^0 .

SES:
$$0 \rightarrow M_{n+1}^{m-1} \xrightarrow{i} \sum \frac{1}{v_n} M_m^n \xrightarrow{v_n} M_m^n \rightarrow 0$$

$\underbrace{i(x) = \frac{x}{v_n}}$

$$\begin{array}{ccccccc} 0 & \rightarrow & M_1^0 & \rightarrow & M_0^1 & \xrightarrow{p} & M_0^1 \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & v_1^{-1}BP_\star/p & \rightarrow & v_1^{-1}BP_\star/p^\infty & \rightarrow & v_1^{-1}BP_\star/p^\infty \rightarrow 0 \\ & & x \mapsto & & \frac{x}{p} & & \end{array}$$

Will use LES for Ext ; $\text{Ext}(v_n^{-1}BP_\star/I_n)$
related to $H_c^*(S(n))$, where " $S(n)$ is a p-adic Lie group."

II. Morava's change of rings

Def $B(n)_\star = v_n^{-1}BP_\star / I_n$
 $\Gamma(n) = B(n)_\star \otimes_{BP_\star} BP_\star \otimes_{BP_\star} B(n)_\star$

$(B(n), \Gamma(n))$ is a Hopf algebroid classifying height n FGLs and isos.

Def $\Sigma(n) = K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_*$

$(K(n)_*, \Sigma(n))$ classifies pushforwards of Honda FGL Γ_n via $K(n)_* \rightarrow R$ and isos of such.

Thm (Morava) For M a $BP_* BP$ -comodule ~~with $I_n M = 0$~~ with $I_n M = 0$ and such that v_n acts invertibly,

$$\begin{aligned} \text{Ext}_{BP_* BP}^s(\Sigma^t BP_*, v_n^{-1} M) &\stackrel{(1)}{\cong} \text{Ext}_{\Gamma(n)}^s(\Sigma^t B(n)_*, B(n)_* \otimes_{BP_*} M) \\ &\stackrel{(2)}{\cong} \text{Ext}_{\Sigma(n)}^s(\Sigma^t K(n)_*, K(n)_* \otimes_{BP_*} M) \end{aligned}$$

(We'll compute $\text{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$.)

Pf idea ① height is an isomorphism invariant

$$\Rightarrow \Gamma(n) \cong BP_* BP \otimes_{BP_*} B(n)_*$$

$$\Rightarrow \Gamma(n) \otimes_{B(n)_*} M \cong BP_* BP \otimes_{BP_*} M$$

$\Rightarrow$ equivalence of categories

$$\left\{ \begin{array}{l} \text{full subcategory of } M \\ \text{satisfying hypotheses} \\ \text{of theorem} \end{array} \right\} \xrightarrow{\sim} \{ \Gamma(n)\text{-comodules} \}$$

② Over $B(n)_*$, two FGLs: one coming from BP_* via quotient (call it F), and Γ_n induced by $K(n)_* \rightarrow B(n)_*$.

Lemma There's a faithfully flat extension
 $B(n)_* \rightarrow C(n)_* \cong K(n)_* \otimes_{B(n)_*} \Gamma(n)$ over which
 $F \cong \Gamma_n$.

Pf of lemma Double complex argument.

TEA

Def $S(n) := \mathbb{F}_p \otimes_{K(n)_*} \Sigma'(n)$ "Morava stabilizer algebra"

Lemma $\text{Hom}(S(n), \mathbb{F}_{p^n}) \cong \mathfrak{S}_n = \text{Aut}(\Gamma_n)$, the
 Morava stabilizer group.

Indeed, $\text{Hom}(S(n), R) = \text{isos. of } \Gamma_n/R$.

Lemma Equivalence of categories

$$\{(K(n)_*, \Sigma'(n))\text{-comodules}\} \longleftrightarrow \{(\mathbb{F}_{p^n}, S(n))\text{-comodules}\}$$

$$\Rightarrow \text{Ext}_{\Sigma'(n)}^{s,t}(K(n)_*, K(n)_* \otimes M) \otimes_{K(n)_*} \mathbb{F}_p \cong \text{Ext}_{S(n)}^s(\mathbb{F}_p, \mathbb{F}_p \otimes_{K(n)_*} M)$$

Lemma $\text{Ext}_{S(n)}^{s,t}(\mathbb{F}_p, \mathbb{F}_p \otimes M) \otimes \mathbb{F}_{p^n} \cong H_c^*(G_n, \mathbb{F}_{p^n} \otimes_{K(n)_*} M)$

Finally, something (kind of) computable:

$$\text{for } n=1, H_c^*(\mathbb{S}_1, \mathbb{F}_p) = \bigwedge_{\mathbb{F}_p} [h_0], |h_0|=1.$$

$$\Rightarrow \text{Ext}_{BP_*BP_*}^{BP_*}(v_i^{-1}BP_*/p) = \bigwedge_{K(i)_*} [h_0], |h_0|=1.$$

III. Calculating $\text{Ext}^i(BP_*)$

So there's a six-term exact sequence:

$$0 \rightarrow \text{Ext}^0(v_i^{-1}BP_*/p) \rightarrow \text{Ext}^0(v_i^{-1}BP_*/p_\infty) \rightarrow \dots \rightarrow \text{Ext}^i(v_i^{-1}BP_*/p_\infty)$$

Lemma $\text{Ext}^0(v_i^{-1}BP_*/p_\infty) \cong \begin{cases} \mathbb{Q}/\mathbb{Z}_{(p)} & \text{in degree } 0 \\ \mathbb{Z}_{(p)}/p^{i+1} & \text{in degree } 2sp^i(p-1) \end{cases} \xrightarrow{0}$

Lemma $\text{Ext}^i(v_i^{-1}BP_*/p_\infty) \cong \begin{cases} \mathbb{Q}/\mathbb{Z}_{(p)} & \text{in deg. } 0 \\ 0 & \text{else} \end{cases}$ by $v_i^{sp^i}/p^{i+1}$, $p \neq 5$, $i \geq 0$.

This is the input for the chromatic SS.

$$E_i^{1,0,*} = \begin{cases} \mathbb{Q}/\mathbb{Z}_{(p)} \\ \mathbb{Z}_{(p)}/p^{i+1} \end{cases} \quad E_i^{0,0,0} = \mathbb{Q}$$

$$E_i^{1,1,0} = \mathbb{Q}/\mathbb{Z}_{(p)}$$

$$\Rightarrow E_\infty^{0,0,0} = \ker(\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$$

$E_2^{1,0,*} = \mathbb{Z}(p)/p^{i+1}$ in positive degrees $= E_\infty^{1,0,*}$.
 This is all that's left: it's the α family.

$$\Sigma^{-1}M, S \rightarrow L, S \rightarrow L_0, S$$

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Lemma $M, S = M(p^\infty)$ (colimit of Moore spaces)

$$\text{Ext}_{E(1)_*E(1)}^{s,t} (E(1)_*, E(1)_*(M(p^\infty))) \Rightarrow \pi_{t-s} \underbrace{L, M(p^\infty)}_{\substack{L, M, S \\ \text{"} \\ M, S}}$$

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$$\text{Ext}_{BP_*BP}^{s,t} (BP_*, v_1^{-1}BP_*/p^\infty)$$