NOTES FOR NU PRE-TALBOT SEMINAR 4/18/13

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As the last speaker of the seminar, it falls to me to say something about topological modular forms. My approach will be contextual; we will try to learn something about tmf by examining the shape of the hole into which it should fit.

1. Genus, Index, and Orientation

1.1. An orientability problem.

Question. Which vector bundles $\pi : V \to X$ are orientable for real K-theory?

One approach to this problem is through the Atiyah-Hirzebruch spectral sequence.[1] We may as well assume that V is orientable for ordinary cohomology, and then the usual Thom isomorphism gives an isomorphism of E_2 pages, which may or may not extend to a map of spectral sequences:

$$\begin{tikzcd} H^p(X;\pi_{-q}ko) \arrow[r, Rightarrow] \arrow[d] & ko^{p+q}(X) \arrow[d, dashed] \\ \widetilde{H}^p(X^V;\pi_{-q}ko) \arrow[r, Rightarrow] & \widetilde{ko}^{p+q}(X). \end{tikzcd}$$

Since this isomorphism is induced by cupping with the Thom classes $\mu_V^{\mathbb{Z}}$ and $\mu_V^{\mathbb{F}_2}$, the Leibniz rule implies that the first obstruction to the existence of the dashed arrow is

$$d_2^{n,-q}\mu_V^{\pi_q ko} = 0,$$

where $n = \dim V$.

Now, one way to construct the AHSS is via the Postnikov tower of ko, in which case the differentials are exactly the k-invariants; in particular, $d_2^{n,-1}$ is a stable cohomology operation $H\mathbb{F}_2 \to \Sigma^2 H\mathbb{F}_2$, and, as one can show, a nontrivial one, whence $d_2^{n,-1} = Sq^2$. Thus our necessary condition for orientability may be rewritten as

$$w_2(V) := Sq^2\mu_V^{\mathbb{F}_2} = 0.$$

[1] For the sake of exposition, I'm going to ride a little roughshod over the distinction between KO and ko.

In other words, we have a nullhomotopy of the composite in the following diagram, and hence a lift to the homotopy fiber:

$$\begin{tikzcd} & BSpin(n) \arrow[d] & \\ X \arrow[ur, dashed] \arrow[r, "V"] & BSO(n) \arrow[r, "w_2"] & K(\mathbb{F}_2, 2). \end{tikzcd}$$

We have proved half of the following

Theorem (Atiyah-Bott-Shapiro)**.** *A vector bundle is KO-orientable if and only if it is a spin bundle.*

Mystery. What does K-theory have to do with spinors?

Clue. If M is a 16-dimensional spin-manifold, the number

$$\int_M 381p_1^4 - 904p_1^2p_2 + 208p_2^2 + 32p_1p_3 - 196p_4$$

is divisible by $464,486,400 = 2^{15}3^45^27$! (That's an exclamation mark, not a factorial symbol...) To say why this is a clue, we need to back up a lot.

1.2. **Manifold invariants.**

Euler characteristic. Perhaps the oldest manifold invariant is the Euler characteristic $\chi(M)$, defined, of course, as the alternating sum of the number of cells in a triangulation of M. Alternatively,

$$\chi(M) = \int_M e(TM),$$

at least if M is orientable of even dimension.[2] In light of these restrictions, we might prefer to work with a less refined invariant, the mod 2 Euler characteristic, for which we always have

$$\chi(M) \bmod 2 = \int_M w_n(TM).$$

Better yet, this version of the invariant defines a *genus* for unoriented cobordism, i.e. a ring homomorphism

$$\chi \bmod 2 : MO_* \to \mathbb{F}_2.$$

[2] As is well known, all manifolds are compact.

Signature. A more interesting invariant is the signature, which is defined only for oriented manifolds of dimension $4k$. Under these hypotheses, Poincaré duality gives us a non-degenerate symmetric bilinear form

$$B_M = \int_M - \wedge - : H^{2k}(M;\mathbb{R}) \to \mathbb{R}$$

and we define the signature $\sigma(M)$ of M to be the signature of B_M.

The following theorem says that there's a similar story to tell about σ as for the mod 2 Euler characteristic:

Theorem (Thom)**.** (1) *The signature is a ring homomorphism*

$$\sigma : MSO_* \to \mathbb{Z}.$$

(2) $[M] = 0 \in MSO_* \otimes \mathbb{Q}$ *if and only if all Pontrjagin numbers of M are zero.*

Putting two and two together, we conclude that there are rational polynomials L_k, homogeneous of degree k, such that

$$\sigma(M) = \int_M L_k(p_1, \ldots, p_k),$$

where $|p_i| = i$. But that's not all: these polynomials are interrelated in some complicated way expressing the fact that σ is a *ring* homomorphism. The words for this are "$\{L_k\}$ is a multiplicative sequence," but we won't have the time to go into what that means; fortunately for us, we have the following

Theorem (Hirzebruch)**.** *There are bijections*

$$\left\{ \begin{array}{c} \textit{ring homomorphisms} \\ \varphi : MSO_* \to \mathbb{Q} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{multiplicative sequences} \\ \textit{of rational polynomials} \end{array} \right\} \leftrightarrow 1 + z\mathbb{Q}[[z]].$$

The proof is essentially the splitting principle.

So if rational genera for MSO_* correspond to invertible power series, which one is σ? The answer is Hirzebruch's famous signature theorem, which we'll give a quick proof of, since it's fun:

Theorem (Hirzebruch)**.** *The signature is the genus determined by the series*

$$L(z) = \frac{\sqrt{z}}{\tanh\sqrt{z}}.$$

Proof. The classes $[\mathbb{CP}^{2k}]$ generate $MSO_* \otimes \mathbb{Q}$, and clearly $\sigma(\mathbb{CP}^{2k}) = 1$, so it suffices to show that the genus determined by L also has this property. Now, since $T\mathbb{CP}^n \oplus \underline{\mathbb{C}} \cong (\gamma_n)^{n+1}$, the total Chern class is given by $c(\mathbb{CP}^n) = (1+z)^{n+1}$, where $z = c_1(\gamma_n)$, and so the total Pontrjagin class is $(1+z^2)^{n+1}$. Hence the recipe for calculating the L-genus is to integrate $L(z^2)^{2k+1}$ over $\mathbb{CP}^{2k}$; this has the effect picking out the coefficient of z^{2k}, which is

$$\begin{aligned}\frac{1}{2\pi i}\oint \frac{dz}{\tanh^{2k+1} z} &= \frac{1}{2\pi i}\oint \frac{du}{u^{2k+1}(1-u^2)} \\ &= \frac{1}{2\pi i}\oint \sum_m \frac{du}{u^{2k-m+1}} \\ &= 1.\end{aligned}$$

□

Note that, modulo knowledge of Pontrjagin numbers, this theorem turns the calculation of the signatures of manifolds into the kind of problem you can give to a computer.

Todd genus. A complex manifold M has a holomorphic Euler characteristic, alias arithmetic genus, defined by

$$\chi(\mathcal{O}_M) = \sum (-1)^i \dim H^i(M; \mathcal{O}_M).$$

In 1937, Todd conjectured that this invariant should be the genus of the multiplicative sequence of polynomials corresponding to the power series

$$\frac{x}{1-e^{-x}}.$$

Proving this to be the case turned out to be hard; Hirzebruch finally did it in 1965, and it was a major motivation behind the Hirzebruch-Riemann-Roch theorem.

It's worth noting that, while the holomorphic Euler characteristic could only be defined for complex manifolds, the power series formula defines the Todd genus for a larger class of manifolds.

$\widehat{A}$-genus. So far the game has been to render geometrically defined invariants algebraic. We now do the opposite.

Given a real vector bundle V, one can complexify and compute the Todd genus $\mathrm{Td}(V \otimes \mathbb{C})$. Since $V \otimes \mathbb{C} \cong V \oplus V$ as real bundles, one might expect the Todd genus to have a "square root." If $\dim V = 2$, the splitting principle allows us to replace $V \otimes \mathbb{C}$ by $L \oplus \overline{L}$ for a complex line bundle L, in which case we find

$$\mathrm{Td}(V \otimes \mathbb{C}) = \mathrm{Td}(L \oplus \overline{L}) = \left(\frac{x}{1-e^{-x}}\right)\left(\frac{-x}{1-e^{x}}\right) = \left(\frac{x/2}{\sinh x/2}\right)^2.$$

Thus we are led to define a new genus for real bundles, the so-called $\widehat{A}$-genus, as the one associated to the power series

$$\frac{x/2}{\sinh x/2}.$$

The polynomial in the Pontrjagin classes that I wrote down earlier is a scaled version of the fourth $\widehat{A}$-polynomial, and the divisibility assertion is a special case of the following 1958 result:

Theorem (Borel-Hirzebruch)**.** *If M is a spin manifold, then $\widehat{A}(M)$ is an integer.*

There is a difference between a theorem and an explanation, and at this point in the story we are missing the later. *Why* should the $\widehat{A}$-genus be an integer on spin manifolds? More generally, we could ask why any of the invariants we've discussed so far are integers.

1.3. **Index theorems.**

Principle. An integer is the dimension of something.

For example, from Hodge theory, we have

$$\begin{aligned}\chi(M) &= \sum(-1)^i \dim H^i(M;\mathbb{R})\\ &= \sum(-1)^i\mathcal{H}^i\\ &= \dim\mathcal{H}^{\text{even}} - \dim\mathcal{H}^{\text{odd}}\\ &= \dim\ker D - \dim\operatorname{coker} D\\ &=: \operatorname{index}(D),\end{aligned}$$

where $\mathcal{H}^i$ denotes the space of harmonic i-forms and D is the Hodge differential $D = d + d^* : \Gamma(\Lambda^{\text{even}}T^*M) \to \Gamma(\Lambda^{\text{odd}}T^*M)$. With a different decomposition of the exterior algebra on T^*M, where $\dim M = 4k$, the operator $d + d^*$ can be made to produce the signature as its index, and complex Hodge theory lets us write the Todd genus of a complex manifold as $\chi(\mathcal{O}_M) = \operatorname{index}(\overline{\partial} + \overline{\partial}^*)$, where $\overline{\partial}$ is the Dolbeault operator. So what about the $\widehat{A}$-genus?

Let M^{4k} be a spin manifold with principal $\mathrm{Spin}(4k)$-bundle $P(M)$. The natural home of $\mathrm{Spin}(n)$ is in the *Clifford algebra*

$$Cl_n := \frac{T(\mathbb{R}^n)}{x^2 = -\|x\|^2}.$$

Alternatively, Cl_n is the $\mathbb{R}$-algebra freely generated by n anticommuting square roots of -1; for example, $Cl_0 \cong \mathbb{R}$, $Cl_1 \cong \mathbb{C}$, and $Cl_2 \cong \mathbb{H}$. Since $\mathrm{Spin}(4k)$ acts on Cl_{4k}, one can form the Clifford bundle $Cl(M)$ of M using the Borel construction, and we obtain a bundle $\$$ of Clifford modules in the same way, starting with an irreducible module. This bundle is equipped with a natural "Dirac" operator $\not{D} : \Gamma(\$^+) \to \Gamma(\$^-)$, which is defined locally by

$$\not{D}\sigma = \sum_j e_j \cdot \nabla_{e_j}\sigma,$$

where $\{e_j\}$ is a local frame, ∇ is the connection, and the dot indicates Clifford multiplication. Regarding this situation, we have the following important

Theorem (Atiyah-Singer)**.**

$$\widehat{A}(M) = \mathrm{index}(\not{D}).$$

The moral of the story so far is that, for the correct class of manifolds, a genus is the index of an operator.

1.4. **Orientations again.**

1	2	3	4	5	6	7	8
$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

This table is really two tables; the first contains the first eight homotopy groups of KO, while the second lists the reduced Grothendieck groups of graded Cl_n-modules for $1 \leq n \leq 8$. We know from Bott that the first repeats with eightfold periodicity, and ABS show that the second does as well. Of course, the two tables are really the same table, and the two periodicities the same periodicity.

If the $\widehat{A}$-genus of a $4k$-dimensional spin manifold is saying something about $4k$-dimensional Clifford modules, this table suggests that there are torsion invariants to be found in other dimensions, and, by considering *families* of operators, Atiyah and Singer found them. The index of a family of operators is now a difference of *vector bundles*, which is to say an element of K-theory, and this fancier index provides a ring homomorphism

$$\widehat{A} : MSpin_* \to KO_*.$$

Returning to homotopy theory, the obvious question is now whether $\widehat{A}$ is the value on homotopy of a map of ring spectra. In other words, coming full circle, is the $\widehat{A}$-genus expressing an orientation of spin bundles for KO?

Construction. Let $V \to X$ be a spin bundle of dimension $8k$.[3] There is a periodicity element β in the Grothendieck group of Cl_8-modules. Let

$$E = P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin(8k)}} \beta^k.$$

Then over the disk bundle $D(V)$ of V, there is a map σ,

$$\begin{tikzcd} \pi^* E^0 \arrow[rr, "\sigma"] \arrow[dr] & & \pi^* E^1 \arrow[dl] \\ & D(V) \arrow[d, "\pi"] & \\ & X, & \end{tikzcd}$$

[3]An orientation is a stable phenomenon, so it is enough to construct a Thom class in these dimensions.

defined by $\sigma_{(p,v)}(x) = v \cdot x$. Since σ restricted to the sphere bundle is an isomorphism, it defines a difference class $[\pi^* E^1, \pi^* E^0, \sigma] \in KO(D(V), S(V)) \cong \widetilde{KO}(X^V)$, which is KO-Thom class for V. The construction is functorial and multiplicative, and so defines a map of ring spectra $MSpin \to KO$ lifting the $\widehat{A}$-genus.

It gets better! This discussion applies in one form or another to all of the manifold invariants we've encountered:[4]

Genus	Operator	Orientation
χ	$d + d^*$	$MO \to \vee_\alpha \Sigma^{n_\alpha} H\mathbb{F}_2$
σ	$d + d^*$	$MSO \to \mathbb{L}$
Td	$\overline{\partial} + \overline{\partial}^*$, $\not{D}^c$	$MU \to MSpin^c \to K$
$\widehat{A}$	$\not{D}$	$MSpin \to KO$

Moral. When you encounter a genus, an operator, or an orientation, you should wonder where the other two are.

2. The Witten Genus

2.1. **Modularity.** In 1988, Witten was led by physics to consider the free loop space LM of a spin manifold M, which is something like an "infinite-dimensional manifold." There's a way to make sense of the statement that LM is *spin*, and it turns out that this happens exactly when the classifying map of TM lifts to $BO\langle 8\rangle =: BString$.

Now, if LM is something like a spin manifold, then it should have a Dirac operator, the index of which should be a genus for string manifolds, valued in the power series ring $\mathbb{Z}[[q]]$ and expressing an orientation of $MString$ for some cohomology theory. One candidate is $KO[[q]]$, essentially since $\widehat{A}$ lifts to KO, but this can't be the right one for a very simple reason. Witten wrote down the characteristic series of this genus and observed that it is the Fourier expansion of a *modular form*; whatever the mystery cohomology theory is, it must have something to do with elliptic curves.

Mystery. Why is the Witten genus of string manifold a modular form?

2.2. **Elliptic cohomology.** Once a point of order 2 is fixed, it turns out that any elliptic curve can be written as

$$y^2 = R(x) = 1 - 2\delta x^2 + \epsilon x^4,$$

[4] A few explanations are in order. First, Thom proved that MO splits as a wedge of suspension of $H\mathbb{F}_2$, so any MO-module spectrum does as well. Second, the natural target for the signature turns out to be something called L-theory, which has to do with quadratic forms and surgery. Third, the Todd genus is most naturally viewed as the complex analogue of the $\widehat{A}$-genus, with its natural home being the bordism of Spin^c-manifolds. Such a manifold carries a Dirac operator that coincides with the Dolbeault operator when the manifold is almost complex.

the so-called Jacobi form. The payoff of this transformation is that the formal group law of such a curve has the beautiful closed form

$$x +_F y = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - \epsilon x^2 y^2}.$$

This FGL was discovered by Euler as an addition formlua for elliptic integrals analogous to the one for the sine function.

This formal group law is defined over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$, and Ochanine noticed that its classifying map factors through a map Φ, which he called the *elliptic genus*:

$$\begin{tikzcd} MU_* \arrow[rr, "F"] \arrow[dr] & & \mathbb{Z}[\frac{1}{2}, \delta, \epsilon] \\ & MSO_* \arrow[ur, "\Phi"'] & \end{tikzcd}$$

With the help of Euler's formula and Landweber's exact functor theorem, Landweber-Ravenel-Stong were able to produce the first elliptic cohomology theory,

$$\mathrm{Ell}_2(X) = MSO_*(X) \otimes_{MSO_*} \mathbb{Z}[\tfrac{1}{2}, \delta, \epsilon, \Delta^{-1}],$$

where $\Delta = \epsilon(\delta^2 - \epsilon)^2$ is the discriminant.

Now, the parameters δ and ϵ are modular forms when viewed as functions on the set of elliptic curves; indeed, the ring of level 2 modular forms is isomorphic to $\mathbb{C}[\delta, \epsilon]$. Later, Hovey showed that the LRS construction lifts to a level 1 theory Ell_1, where the corresponding ring is $\mathbb{C}[E_4, E_6]$, and there are many more brands of elliptic cohomology. Unfortunately, although the scent is clearly in the air, none of these theories manages to reproduce the Witten genus.

2.3. **The σ-orientation.** The brilliant idea of Hopkins and his collaborators is to treat this cohomology theory as the mysterious object that it is. They propose to study maps of ring spectra $MString \to E$ in general, and to try to "solve for E."

There are two simplifications that make this problem tractable; we take E to be even and periodic, and we replace $MString$ by $MU\langle 6\rangle$, since $MString$ is rather mysterious.[5] These adjustments place us within arm's reach of the nexus between algebraic topology and algebraic geometry that is the theory of complex-oriented cohomology theories, and the approach to the problem will be very algebro-geometric. After a little manipulation, we have

$$\mathrm{RingSpectra}(MU\langle 6\rangle, E) \cong \operatorname{Spec} E_0 MU\langle 6\rangle(\pi_0 E),$$

[5]The following diagram is useful to bear in mind:

$$\begin{array}{ccccc} BU\langle 6\rangle & \longrightarrow & BSU & \longrightarrow & BU \\ \downarrow & & \downarrow & & \downarrow \\ BString & \longrightarrow & BSpin & \longrightarrow & BSO. \end{array}$$

so our task is to understand the functor $\operatorname{Spec} E_0 MU\langle 6\rangle$, which should be the same as understanding the functor $\operatorname{Spec} E_0 BU\langle 6\rangle$ together with the Thom isomorphism. We'll undertake a very brief sketch of the argument, so as not to get bogged down in the algebraic geometry.

Just as everything that we know about BU is obtained from the map $\mathbb{CP}^\infty \to BU$ classifying the tautological line bundle, we should understand $BU\langle 6\rangle$ via the map

$$(\mathbb{CP}^\infty)^3 \xrightarrow{\otimes(1-L_i)} BU\langle 6\rangle.$$

This defines an element f of the set obtained by evaluating

$$\operatorname{Spec} E_0(\mathbb{CP}^\infty)^3 \cong \underline{\operatorname{Hom}}((\operatorname{Spf} E^0\mathbb{CP}^\infty)^3, \mathbb{G}_m)$$

at $E_0BU\langle 6\rangle$ (this isomorphism is called *Cartier duality*). This f is a function between group schemes, and, while it's not a homomorphism, it inherits the symmetries of the topological map, which are encapsulated by saying that f is a "rigid, symmetric 2-cocycle." As such, it determines a map from $\operatorname{Spec} E_0BU\langle 6\rangle$ to the functor representing rigid, symmetric 2-cocycles, which, after a great deal of work, can be shown to be an isomorphism.

To summarize, $E_0BU\langle 6\rangle$ *represents certain rigid functions on the formal group* $(\operatorname{Spf} E^0\mathbb{CP}^\infty)^3$.

Now, if $V \to X$ is a vector bundle, then $\widetilde{E}^0 X^V$ is an E^0X-module of rank 1; therefore, from the point of view of algebraic geometry, the Thom isomorphism is saying something about the relationship between *functions* and *sections of a line bundle.* In fact, there is a functor $\mathbb{L}$ from vector bundles on X to line bundles on $\operatorname{Spf} E^0X$, and with this functor in hand the result about $BU\rangle 6\rangle$ implies that $E_0MU\langle 6\rangle$ *represents certain rigid sections of* $\mathbb{L}(\otimes(1-L_i))$.

What's good about this result is that the theorem of the cube, a big gun from classical algebraic geometry, now implies that there is only one such section, provided $\operatorname{Spf}\mathbb{CP}^\infty$ is the formal completion of an elliptic curve.

Definition. An *elliptic spectrum* is a triple (E, C, φ) with

(1) E an even periodic ring spectrum,
(2) C an elliptic curve over $\pi_0 E$, and
(3) $\varphi : \operatorname{Spf} E^0\mathbb{CP}^\infty \to \widehat{C}$ an isomorphism.

A map of elliptic spectra is a map of ring spectra $f : E_1 \to E_2$ and an isomorphism $f^\# : (\pi_0 f)^* C_1 \to C_2$.

To summarize the above discussion,

Theorem (Ando-Hopkins-Strickland)**.** *If (E, C, φ) is an elliptic spectrum, there is a canonical map*

$$\sigma_E : MU\langle 6\rangle \to E$$

that is natural for maps of elliptic spectra.

2.4. **Modularity again.** At this point, we're desperately in need of some examples!

Cusps. The cuspidal curve C_a defined over $\operatorname{Spec}\mathbb{Z}$ by the equation $y^2z = x^3$ is a group away from the cusp, and there is an isomorphism

$$\alpha : \widehat{C}_a \cong \widehat{\mathbb{G}}_a.$$

Since the formal group law of ordinary periodic cohomology HP is the additive law, we see that (HP, C_a, α) as an elliptic spectrum. The σ-orientation in this case expresses the familiar fact that a bundle whose structure group lifts to $U\langle 6\rangle$ is orientable.

Nodes. The nodal cubic C_m defined over $\operatorname{Spec}\mathbb{Z}$ by the equation $y^2z + xyz = x^3$ is similarly a group away from the node, and there is an isomorphism

$$\beta : \widehat{C}_m \cong \widehat{\mathbb{G}}_m.$$

Since the formal group law of complex K-theory is the multiplicative law, $(K, C_m, \hat{\beta})$ as an elliptic spectrum. Here the σ-orientation is the restriction to $MU\langle 6\rangle$ of the K-theory orientation of Spin^c-bundles, which we saw was a lift to spectra of the Todd genus.

Jacobi. In this setting, the result of Landweber-Ravenel-Stong that the Jacobi formal group law is Landweber exact is the statement that there is an even, periodic ring spectrum Ell_2 and an isomorphism $\gamma : \widehat{C}_J \cong \operatorname{Spf}\mathrm{Ell}_2^0\mathbb{CP}^\infty$, where C_J is the Jacobi quartic defined over $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$ by $y^2 = 1 - 2\delta x^2 + \epsilon x^4$. Thus $(\mathrm{Ell}_2, C_J, \gamma)$ is an elliptic spectrum, and the σ-orientation is a lift of Ochanine's elliptic genus. In the same way, level 1 elliptic cohomology is naturally an elliptic spectrum and so receives a σ-orientation.

Weierstrass. We work over $\operatorname{Spec}\mathbb{C}$ with the curve $C_\Lambda = \mathbb{C}/\Lambda$. The projection $\mathbb{C} \to \mathbb{C}/\Lambda$ induces an isomorphism

$$\varphi_\Lambda : \widehat{C}_\Lambda \cong \widehat{\mathbb{C}} = \operatorname{Spf} E_\Lambda^0\mathbb{CP}^\infty,$$

where E_Λ represents $H^*(-, \mathbb{C}[u_\Lambda^\pm])$ and $|u_\Lambda| = 2$. Thinking in terms of genera, the σ-orientation for the elliptic spectrum $(E_\Lambda, C_\Lambda, \varphi_\Lambda)$ assigns to a $2n$-dimensional $MU\langle 6\rangle$-manifold M a complex number $\Phi(M; \Lambda)$ via

$$(\sigma_{E_\Lambda})_*([M]) = \Phi(M; \Lambda)u_\Lambda^{2n}.$$

Now, if $\mathcal{I}(0)$ denotes the ideal sheaf of the zero section of $\operatorname{Spf} E_\Lambda^0\mathbb{CP}^\infty$, then $\mathcal{I}(0) \cong \ker(E_\Lambda^0\mathbb{CP}^\infty \to E_\Lambda^0) = \widetilde{E}_\Lambda^0\mathbb{CP}^\infty$, so

$$\omega := \mathcal{I}(0)/\mathcal{I}(0)^2 \cong \widetilde{E}_\Lambda^0 S^2 = \pi_2 E_\Lambda,$$

and we may take $u_\Lambda = dz$, the invariant differential on C_Λ. For $\gamma \in SL_2(\mathbb{Z})$, there is an isomorphism $C_{\Lambda(\tau)} \cong C_{\Lambda(\gamma\tau)}$ of the form

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}, \qquad dz \mapsto \frac{dz}{c\tau + d},$$

so that, by the naturality of the σ-orientation, we have that

$$\Phi(M; \Lambda(\gamma\tau)) = (c\tau + d)^{-n}\Phi(M; \Lambda(\tau)).$$

This is extremely suggestive, but more can be said. Each of the curves C_Λ is the geometric fiber of a generalized elliptic curve C over $\mathcal{O}_{\mathbb{H}}$, the ring of holomorphic functions on the upper half-plane, defined by the equation

$$y^2z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3.$$

If E is the spectrum representing the cohomology theory $H^*(-; \mathcal{O}_{\mathbb{H}}[u^{\pm}])$, by naturality the σ-orientation factors as

$$MU\langle 6\rangle \to E \to E_{\Lambda(\tau)}$$

for every τ, which shows that $\Phi(M; \Lambda(\tau)) \in \mathcal{O}_{\mathbb{H}}$ when considered as a function of τ. In other words, the genus corresponding to the σ-orientation for this family is naturally valued in modular forms!

Tate. We have seen that the σ-orientation exhibits a natural modularity. As the source of the modularity of the Witten genus was the big mystery that we began with, this is a great success.

Now, the Witten genus is valued in power series that are Fourier expansions of modular forms, so, in order to recover the Witten genus explicitly, the natural thing to do is to try to "take the Fourier expansion" of the previous example. This amounts to considering the map $\mathbb{H} \to \mathbb{D}$ given by $\tau \mapsto e^{2\pi i\tau}$, corresponding to the reparametrization of the Weierstrass curve as

$$\frac{\mathbb{C}^\times}{u \sim qu}, \quad q \in \mathring{\mathbb{D}}.$$

One has a corresponding generalized elliptic curve over $\mathcal{O}_{\mathbb{D}}$. It is given explicitly by $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. In fact, the power series expansions of a_4 and a_6 have integer coefficients, so that this same equation defines an elliptic curve C_{Tate} over $\operatorname{Spec}\mathbb{Z}[[q]]$. Since the projection $\rho : \mathbb{C}^\times \to C_q$ induces an isomorphism $\hat{\rho} : \widehat{C}_q \cong \mathbb{G}_m$, we see that

$$(K[[q]], C_{\mathrm{Tate}}, \hat{\rho}_{\mathrm{Tate}})$$

is an elliptic spectrum, and essentially the same argument as before shows that the σ-orientation for this theory is valued in (Fourier expansions of) modular forms. Moreover, we have the

Theorem (Ando-Hopkins-Strickland)**.** *The diagram*

$$\begin{tikzcd}
MU\langle 6\rangle \arrow[r, "\sigma"] \arrow[d] & K[[q]] \\
MString \arrow[r, "w"] & KO[[q]], \arrow[u, "\otimes\mathbb{C}"']
\end{tikzcd}$$

where w denotes the Witten genus.

In other words, the σ-orientation essentially reproduces the Witten genus and explains its mysterious modularity.

2.5. **Topological modular forms.**

One might object at this point that we haven't really accomplished what we set out to do, which was to find *the* cohomology theory for which the Witten genus expressed an orientation; instead, we have found many cohomology theories, each seemingly as good as the next.

Of course, we are tempted to think of σ as a cone on a big diagram of elliptic spectra:

$$\begin{tikzcd}
 & & & \mathrm{Ell}_1 \arrow[r] & \mathrm{Ell}_2 & \\
H\mathbb{C}[u_\Lambda^\pm] & & & & & \\
 & H\mathcal{O}_{\mathbb{H}}[u^\pm] \arrow[ul] \arrow[dl] & & MU\langle 6\rangle \arrow[l] \arrow[uu] \arrow[uur] \arrow[ulll] \arrow[rr] \arrow[dlll] \arrow[ddll] \arrow[dd] \arrow[drr] & & HP \\
H\mathbb{C}[u_{\Lambda'}^\pm] & & & & & K \arrow[u] \\
 & H\mathcal{O}_{\mathbb{D}}[u^\pm] \arrow[uu] & & K[[q]] \arrow[ll] \arrow[urr] & &
\end{tikzcd}$$

Taking the limit of this diagram, we should obtain the right theory.

This is much easier said than done, but it can be done after a great deal of work:

Theorem (Goerss-Hopkins-Miller)**.** *There is a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-ring spectra on the étale site of the moduli stack of elliptic curves.*

The spectrum of topological modular forms is then taken to be the derived global sections of this sheaf,

$$tmf = \mathbb{R}\Gamma(\mathcal{O}^{\mathrm{top}}).$$

Of course, once one has this spectrum, one wants to know that one has the desired orientation, and it is shown in [1] that the Witten genus lifts to a map of E_∞-ring spectra

$$w : MString \to tmf.$$

I'll close by noting that the technique of that argument is to characterize the set of homotopy classes of such E_∞-ring maps in terms of *characteristic power series*. Hirzebruch would be proud!

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